Contents

Preface

This solutions manual is designed to accompany the tenth edition of *Linear Algebra with Applications* by Steven J. Leon and Lisette de Pillis. The manual contains the complete solutions to all of the nonroutine exercises and Chapter test questions in the first seven chapters the book. Each of those chapters also includes a set of MATLAB computer exercises. Most of the MATLAB computations are straightforward. and consequently the computational results are not included in this manual. However, the MATLAB Exercises also include questions related to the computations. The purpose of the questions is to emphasize the significance of the computations. This manual does provide the answers to most of these questions.

Chapter 1

Matrices and Systems of Equations

1 SYSTEMS OF LINEAR EQUATIONS

1. (a) The solution is $(4,\ 3)$.
(b) The solution is $(1,\ 2,\ 7)$.
(c) The solution is $(1,\ 0,\ -1,\ 2)$.
(d) The solution is $\left(\frac{1}{2},\ -\frac{1}{2},\ \frac{1}{4},\ \frac{1}{3},\ 0\right)$.

2. (a) $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 7 & -1 & 2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 4 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 & 16 & 3 & 1 \\ 0 & 4 & 4 & 6 & 3 \\ 0 & 0 & -8 & 27 & -7 \\ 0 & 0 & 0 & 3 & 11 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

5. (a)
$$\begin{aligned} 3x_1 \qquad &= 6 \\ 2x_2 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - x_2 + 5x_3 &= 8 \\ 3x_1 \qquad + 2x_3 &= 0 \end{aligned}$$

(c)
$$\begin{aligned} x_1 - 2x_2 + x_3 &= 4 \\ 7x_1 \qquad + 5x_3 &= 2 \\ -3x_1 + 2x_2 \qquad &= 0 \end{aligned}$$

(d)
$$\begin{aligned} x_1 - 2x_2 \qquad - 8x_4 &= 5 \\ 2x_1 + x_2 + 3x_3 + 4x_4 &= 6 \\ -3x_2 + x_3 - x_4 &= 7 \\ 8x_1 + 4x_2 + x_3 + x_4 &= 9 \end{aligned}$$

6. (a) The solution is $(5,\ -6)$.
(b) The solution is $(3,\ 7)$.
(c) The solution is $\left(\frac{3}{7},\ \frac{4}{7}\right)$.
(d) The solution is $(1,\ -2,\ 3)$.
(e) The solution is $(-4,\ 2,\ 5)$.
(f) The solution is $(1,\ 2,\ -1)$.
(g) The solution is $(0,\ 1,\ 1)$.
(h) The solution is $(1,\ 2,\ 3,\ -4)$.

7. The solutions are $(2,\ 3)$ and $(3,\ 2)$.

8. The solutions are $(1,\ 2,\ -1)$ and $(2,\ 3,\ -1)$.

9. Given the system

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ -m_2x_1 + x_2 &= b_2 \end{aligned}$$

one can eliminate the variable x_2 by subtracting the first row from the second. One then obtains the equivalent system

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ (m_1 - m_2)x_1 &= b_2 - b_1 \end{aligned}$$

(a) If $m_1 \neq m_2$, then one can solve the second equation for x_1

$$x_1 = \frac{b_2 - b_1}{m_1 - m_2}$$

One can then plug this value of x_1 into the first equation and solve for x_2. Thus, if $m_1 \neq m_2$, there will be a unique ordered pair (x_1, x_2) that satisfies the two equations.

(b) If $m_1 = m_2$, then the x_1 term drops out in the second equation

$$0 = b_2 - b_1$$

This is possible if and only if $b_1 = b_2$.

(c) If $m_1 \neq m_2$, then the two equations represent lines in the plane with different slopes. Two nonparallel lines intersect in a point. That point will be the unique solution to the system. If $m_1 = m_2$ and $b_1 = b_2$, then both equations represent the same line and consequently every point on that line will satisfy both equations. If $m_1 = m_2$ and $b_1 \neq b_2$, then the equations represent parallel lines. Since parallel lines do not intersect, there is no point on both lines and hence no solution to the system.

10. The system must be consistent since $(0, 0)$ is a solution.

11. A linear equation in 3 unknowns represents a plane in three space. The solution set to a 3×3 linear system would be the set of all points that lie on all three planes. If the planes are parallel or one plane is parallel to the line of intersection of the other two, then the solution set will be empty. The three equations could represent the same plane or the three planes could all intersect in a line. In either case the solution set will contain infinitely many points. If the three planes intersect in a point, then the solution set will contain only that point.

2 ROW ECHELON FORM

2. (a) The solution is $(10,\ 3)$.
(b) The system is inconsistent.
(c) The solution is $(3,\ 0,\ -2)$.
(d) The solution set consists of all ordered triples of the form $(-2\alpha - 9,\ \alpha,\ 3)$.
(e) The system is inconsistent.
(f) The solution is $(0,\ 0,\ 2)$.

3. (a) The solution is $(3,\ -2,\ 5)$.
(b) The solution set consists of all ordered triples of the form $(\alpha,\ -3,\ 15)$.
(c) The system is inconsistent.
(d) The solution set consists of all ordered triples of the form $(2\alpha + 5,\ \alpha,\ -1)$.
(e) The solution set consists of all ordered quadruples of the form $(6\alpha + 5\beta,\ \alpha,\ -3\beta - 6,\ \beta)$.
(f) The solution set consists of all ordered quadruples of the form $(-2\alpha - \beta,\ \alpha,\ \beta,\ 3)$.

4. (a) x_1, x_2, and x_3 are lead variables.
(b) x_2 and x_3 are lead variables, and x_1 is a free variable.
(c) x_1 and x_2 are lead variables, and x_3 is a free variable.
(d) x_1 and x_3 are lead variables, and x_2 is a free variable.
(e) x_1 and x_3 are lead variables, and x_2 and x_4 are free variables.
(f) x_1 and x_4 are lead variables, and x_2 and x_3 are free variables.

5. (a) The solution is $(7,\ -3)$.
(b) The system is inconsistent.
(c) The solution is $(0,\ 0)$.
(d) The solution set consists of all ordered triples of the form $\left(\frac{8}{5}\alpha + \frac{9}{5},\ -\frac{1}{5}\alpha + \frac{2}{5},\ \alpha\right)$.
(e) The solution set consists of all ordered triples of the form $\left(-\frac{7}{11}\alpha - 1,\ \frac{1}{11}\alpha + 2,\ \alpha\right)$.
(f) The system is inconsistent.
(g) The solution is $\left(-3,\ 4,\ -\frac{1}{2},\ -\frac{1}{2}\right)$.
(h) The system is inconsistent.
(i) The solution is $(-1,\ 2,\ -1)$.
(j) The solution set consists of all ordered quadruples of the form $\left(\alpha + 4,\ \frac{4}{5}\alpha + \frac{2}{5},\ -\frac{2}{5}\alpha + \frac{4}{5},\ \alpha\right)$.
(k) The solution set consists of all ordered quadruples of the form $(\alpha - 9,\ -2\alpha + 12,\ \alpha,\ 0)$.
(l) The solution set consists of all ordered triples of the form $\left(-\frac{5}{3}\alpha + \frac{4}{3},\ \frac{4}{3}\alpha + \frac{1}{3},\ \alpha\right)$.

6. (a) The solution is $(1,\ -1)$.
(b) The solution set consists of all ordered quadruples of the form $(1,\ 4-\alpha,\ -1,\ \alpha)$.
(c) The solution set consists of all ordered triples of the form $\left(-\frac{15}{11}\alpha+\frac{6}{11},\ \frac{4}{11}\alpha+\frac{5}{11},\ \alpha\right)$.
(d) The solution set consists of all ordered quadruples of the form $\left(-\frac{14}{25}\alpha+\frac{24}{25},\ -\frac{2}{25}\alpha+\frac{7}{25},\ -\frac{1}{5}\alpha+\frac{6}{5},\ \alpha\right)$.

7. A homogeneous linear equation in 3 unknowns corresponds to a plane that passes through the origin in 3-space. Two such equations would correspond to two planes through the origin. If one equation is a multiple of the other, then both represent the same plane through the origin and every point on that plane will be a solution to the system. If one equation is not a multiple of the other, then we have two distinct planes that intersect in a line through the origin. Every point on the line of intersection will be a solution to the linear system. So in either case the system must have infinitely many solutions.

In the case of a nonhomogeneous 2×3 linear system, the equations correspond to planes that do not both pass through the origin. If one equation is a multiple of the other, then both represent the same plane and there are infinitely many solutions. If the equations represent planes that are parallel, then they do not intersect and hence the system will not have any solutions. If the equations represent distinct planes that are not parallel, then they must intersect in a line and hence there will be infinitely many solutions. So the only possibilities for a nonhomogeneous 2×3 linear system are 0 or infinitely many solutions.

8. Using Gauss–Jordan reduction to solve the system, we see that any real number $a \neq -4$ will give a unique solution.

9. (a) Since the system is homogeneous it must be consistent.

13. A homogeneous system is always consistent since it has the trivial solution $(0,\dots,0)$. If the reduced row echelon form of the coefficient matrix involves free variables, then there will be infinitely many solutions. If there are no free variables, then the trivial solution will be the only solution.

14. A nonhomogeneous system could be inconsistent in which case there would be no solutions. If the system is consistent and underdetermined, then there will be free variables and this would imply that we will have infinitely many solutions.

16. At each intersection, the number of vehicles entering must equal the number of vehicles leaving in order for the traffic to flow. This condition leads to the following system of equations

$$\begin{aligned} x_1+a_1 &= x_2+b_1\\ x_2+a_2 &= x_3+b_2\\ x_3+a_3 &= x_4+b_3\\ x_4+a_4 &= x_1+b_4 \end{aligned}$$

If we add all four equations, we get

$$x_1+x_2+x_3+x_4+a_1+a_2+a_3+a_4 = x_1+x_2+x_3+x_4+b_1+b_2+b_3+b_4$$

and hence

$$a_1+a_2+a_3+a_4 = b_1+b_2+b_3+b_4$$

17. If (c_1, c_2) is a solution, then

$$\begin{aligned} a_{11}c_1+a_{12}c_2 &= 0\\ a_{21}c_1+a_{22}c_2 &= 0 \end{aligned}$$

Multiplying both equations through by α, one obtains

$$\begin{aligned} a_{11}(\alpha c_1)+a_{12}(\alpha c_2) &= \alpha\cdot 0 = 0\\ a_{21}(\alpha c_1)+a_{22}(\alpha c_2) &= \alpha\cdot 0 = 0 \end{aligned}$$

Thus $(\alpha c_1, \alpha c_2)$ is also a solution.

18. (a) If $x_4 = 0$, then x_1, x_2, and x_3 will all be 0. Thus if no glucose is produced, then there is no reaction. $(0, 0, 0, 0)$ is the trivial solution in the sense that if there are no molecules of carbon dioxide and water, then there will be no reaction.
(b) If we choose another value of x_4, say $x_4 = 2$, then we end up with solution $x_1 = 12$, $x_2 = 12$, $x_3 = 12$, $x_4 = 2$. Note the ratios are still 6:6:6:1.

3 MATRIX ARITHMETIC

1. (e) $\begin{pmatrix} 8 & -15 & 11 \\ 0 & -4 & -3 \\ -1 & -6 & 6 \end{pmatrix}$

(g) $\begin{pmatrix} 5 & -10 & 15 \\ 5 & -1 & 4 \\ 8 & -9 & 6 \end{pmatrix}$

2. (d) $\begin{pmatrix} 36 & 10 & 56 \\ 10 & 3 & 16 \end{pmatrix}$

4. (a) $\begin{pmatrix} 1 & -2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ 3 & -7 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

5. (a) $5A + 2A = \begin{pmatrix} 5 & 25 \\ 0 & 35 \\ 10 & 20 \end{pmatrix} + \begin{pmatrix} 2 & 10 \\ 0 & 14 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 7 & 35 \\ 0 & 49 \\ 14 & 28 \end{pmatrix}$

$7A = \begin{pmatrix} 7 & 35 \\ 0 & 49 \\ 14 & 28 \end{pmatrix}$

(b) $4\,(2A) = 4\begin{pmatrix} 2 & 10 \\ 0 & 14 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 8 & 40 \\ 0 & 56 \\ 16 & 32 \end{pmatrix}$

$$8A = \begin{pmatrix} 8 & 40 \\ 0 & 56 \\ 16 & 32 \end{pmatrix}$$

(c) $A^T = \begin{pmatrix} 1 & 0 & 2 \\ 5 & 7 & 4 \end{pmatrix}$

$$(A^T)^T = \begin{pmatrix} 1 & 0 & 2 \\ 5 & 7 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 5 \\ 0 & 7 \\ 2 & 4 \end{pmatrix} = A$$

6. (a) $A + B = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 5 & 3 \end{pmatrix} = B + A$

(b) $2(A + B) = 2\begin{pmatrix} 0 & 0 & 1 \\ -1 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ -2 & 10 & 6 \end{pmatrix}$

$$2A + 2B = \begin{pmatrix} 2 & -4 & 8 \\ 0 & 2 & 6 \end{pmatrix} + \begin{pmatrix} -2 & 4 & -6 \\ -2 & 8 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ -2 & 10 & 6 \end{pmatrix}$$

(c) $(A + B)^T = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 5 & 3 \end{pmatrix}^T = \begin{pmatrix} 0 & -1 \\ 0 & 5 \\ 1 & 3 \end{pmatrix}$

$$A^T + B^T = \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 2 & 4 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 5 \\ 1 & 3 \end{pmatrix}$$

7. (a) $-2(AB) = -2\begin{pmatrix} 2 & -13 \\ 10 & -16 \\ -4 & -9 \end{pmatrix} = \begin{pmatrix} -4 & 26 \\ -20 & 32 \\ 8 & 18 \end{pmatrix}$

$$(-2A)B = \begin{pmatrix} -8 & 2 \\ -12 & -4 \\ -4 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 26 \\ -20 & 32 \\ 8 & 18 \end{pmatrix}$$

$$A(-2B) = \begin{pmatrix} 4 & -1 \\ 6 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} -2 & 6 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} -4 & 26 \\ -20 & 32 \\ 8 & 18 \end{pmatrix}$$

(b) $(AB)^T = \begin{pmatrix} 2 & -13 \\ 10 & -16 \\ -4 & -9 \end{pmatrix}^T = \begin{pmatrix} 2 & 10 & -4 \\ -13 & -16 & -9 \end{pmatrix}$

$$B^TA^T = \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 4 & 6 & 2 \\ -1 & 2 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 10 & -4 \\ -13 & -16 & -9 \end{pmatrix}$$

8. (a) $(A+B)+C = \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 5 & 7 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 7 \\ 7 & 5 \end{pmatrix}$

$A+(B+C) = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 8 & 5 \\ 7 & 6 \end{pmatrix} = \begin{pmatrix} 9 & 7 \\ 7 & 5 \end{pmatrix}$

(b) $(AB)C = \begin{pmatrix} 5 & 2 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 5 & 7 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 37 & 43 \\ -17 & -15 \end{pmatrix}$

$A(BC) = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 13 \\ 17 & 15 \end{pmatrix} = \begin{pmatrix} 37 & 43 \\ -17 & -15 \end{pmatrix}$

(c) $A(B+C) = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 8 & 5 \\ 7 & 6 \end{pmatrix} = \begin{pmatrix} 22 & 17 \\ -7 & -6 \end{pmatrix}$

$AB+AC = \begin{pmatrix} 5 & 2 \\ -1 & -2 \end{pmatrix} + \begin{pmatrix} 17 & 15 \\ -6 & -4 \end{pmatrix} = \begin{pmatrix} 22 & 17 \\ -7 & -6 \end{pmatrix}$

(d) $(A+B)C = \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 7 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 20 & 28 \\ 11 & 11 \end{pmatrix}$

$AC+BC = \begin{pmatrix} 17 & 15 \\ -6 & -4 \end{pmatrix} + \begin{pmatrix} 3 & 13 \\ 17 & 15 \end{pmatrix} = \begin{pmatrix} 20 & 28 \\ 11 & 11 \end{pmatrix}$

9. (a) $\mathbf{b} = 2\mathbf{a_1} + 3\mathbf{a_2}$.

(b) $\mathbf{x} = (2,\ 3)^T$ is a solution since $\mathbf{b} = 2\mathbf{a_1} + 3\mathbf{a_2}$. There are no other solutions since the echelon form of A is strictly triangular.

(c) The solution to $A\mathbf{x} = \mathbf{c}$ is $\mathbf{x} = \left(\frac{9}{5},\ \frac{7}{5}\right)^T$. Therefore, $\mathbf{c} = \frac{9}{5}\mathbf{a_1} + \frac{7}{5}\mathbf{a_2}$.

11. The given information implies that

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

are both solutions to the system. So the system is consistent and since there is more than one solution, the row echelon form of A must involve a free variable. A consistent system with a free variable has infinitely many solutions.

12. The system is consistent since $\mathbf{x} = (1, 1, 1, 1)^T$ is a solution. The system can have at most 3 lead variables since A only has 3 rows. Therefore, there must be at least one free variable. A consistent system with a free variable has infinitely many solutions.

13. (a) It follows from the reduced row echelon form that the free variables are x_2, x_4, x_5. If we set $x_2 = a$, $x_4 = b$, $x_5 = c$, then

$$\begin{aligned} x_1 &= -2 - 2a - 3b - c \\ x_3 &= 5 - 2b - 4c \end{aligned}$$

and hence the solution consists of all vectors of the form

$$\mathbf{x} = (-2 - 2a - 3b - c,\ a,\ 5 - 2b - 4c,\ b,\ c)^T$$

(b) If we set the free variables equal to 0, then $\mathbf{x}_0 = (-2, 0, 5, 0, 0)^T$ is a solution to $A\mathbf{x} = \mathbf{b}$ and hence

$$\mathbf{b} = A\mathbf{x}_0 = -2\mathbf{a}_1 + 5\mathbf{a}_3 = (8, -7, -1, 7)^T$$

14. If w_3 is the weight given to professional activities, then the weights for research and teaching should be $w_1 = 3w_3$ and $w_2 = 2w_3$. Note that

$$1.5w_2 = 3w_3 = w_1,$$

so the weight given to research is 1.5 times the weight given to teaching. Since the weights must all add up to 1, we have

$$1 = w_1 + w_2 + w_3 = 3w_3 + 2w_3 + w_3 = 6w_3$$

and hence it follows that $w_3 = \frac{1}{6}$, $w_2 = \frac{1}{3}$, $w_1 = \frac{1}{2}$. If C is the matrix in the example problem from the Analytic Hierarchy Process Application, then the rating vector $\mathbf{r}$ is computed by multiplying C times the weight vector $\mathbf{w}$.

$$\mathbf{r} = C\mathbf{w} = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{10} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{43}{120} \\ \frac{45}{120} \\ \frac{32}{120} \end{pmatrix}$$

15. A^T is an $n \times m$ matrix. Since A^T has m columns and A has m rows, the multiplication A^TA is possible. The multiplication AA^T is possible since A has n columns and A^T has n rows.

16. If A is skew-symmetric, then $A^T = -A$. Since the (j, j) entry of A^T is a_{jj} and the (j, j) entry of $-A$ is $-a_{jj}$, it follows that $a_{jj} = -a_{jj}$ for each j and hence the diagonal entries of A must all be 0.

17. The search vector is $\mathbf{x} = (1, 0, 1, 0, 1, 0)^T$. The search result is given by the vector

$$\mathbf{y} = A^T\mathbf{x} = (1, 2, 2, 1, 1, 2, 1)^T$$

The ith entry of $\mathbf{y}$ is equal to the number of search words in the title of the ith book.

18. If $\alpha = a_{21}/a_{11}$, then

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & b \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ \alpha a_{11} & \alpha a_{12} + b \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & \alpha a_{12} + b \end{pmatrix}$$

The product will equal A provided

$$\alpha a_{12} + b = a_{22}$$

Thus we must choose

$$b = a_{22} - \alpha a_{12} = a_{22} - \frac{a_{21}a_{12}}{a_{11}}$$

4 MATRIX ALGEBRA

1. (a) $(A + B)^2 = (A + B)(A + B) = (A + B)A + (A + B)B = A^2 + BA + AB + B^2$
For real numbers, $ab + ba = 2ab$; however, with matrices $AB + BA$ is generally not equal to $2AB$.

(b)

$$\begin{aligned}(A+B)(A-B) &= (A+B)(A-B)\\ &= (A+B)A-(A+B)B\\ &= A^2+BA-AB-B^2\end{aligned}$$

For real numbers, $ab - ba = 0$; however, with matrices $AB - BA$ is generally not equal to O.

2. If we replace a by A and b by the identity matrix, I, then both rules will work, since

$$(A+I)^2 = A^2 + IA + AI + B^2 = A^2 + AI + AI + B^2 = A^2 + 2AI + B^2$$

and

$$(A+I)(A-I) = A^2 + IA - AI - I^2 = A^2 + A - A - I^2 = A^2 - I^2$$

3. There are many possible choices for A and B. For example, one could choose

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

More generally if

$$A = \begin{pmatrix} a & b \\ ca & cb \end{pmatrix} \qquad B = \begin{pmatrix} db & eb \\ -da & -ea \end{pmatrix}$$

then $AB = O$ for any choice of the scalars a, b, c, d, e.

4. To construct nonzero matrices A, B, C with the desired properties, first find nonzero matrices C and D such that $DC = O$ (see Exercise 3). Next, for any nonzero matrix A, set $B = A + D$. It follows that

$$BC = (A+D)C = AC + DC = AC + O = AC$$

5. A 2×2 symmetric matrix is one of the form

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Thus

$$A^2 = \begin{pmatrix} a^2+b^2 & ab+bc \\ ab+bc & b^2+c^2 \end{pmatrix}$$

If $A^2 = O$, then its diagonal entries must be 0.

$$a^2 + b^2 = 0 \qquad \text{and} \qquad b^2 + c^2 = 0$$

Thus $a = b = c = 0$ and hence $A = O$.

6. Let

$$D = (AB)C = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

It follows that

$$\begin{aligned} d_{11} &= (a_{11}b_{11}+a_{12}b_{21})c_{11} + (a_{11}b_{12}+a_{12}b_{22})c_{21}\\ &= a_{11}b_{11}c_{11} + a_{12}b_{21}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{22}c_{21}\\ d_{12} &= (a_{11}b_{11}+a_{12}b_{21})c_{12} + (a_{11}b_{12}+a_{12}b_{22})c_{22}\\ &= a_{11}b_{11}c_{12} + a_{12}b_{21}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{22}c_{22}\end{aligned}$$

$$\begin{aligned} d_{21} &= (a_{21}b_{11} + a_{22}b_{21})c_{11} + (a_{21}b_{12} + a_{22}b_{22})c_{21} \\ &= a_{21}b_{11}c_{11} + a_{22}b_{21}c_{11} + a_{21}b_{12}c_{21} + a_{22}b_{22}c_{21} \\ d_{22} &= (a_{21}b_{11} + a_{22}b_{21})c_{12} + (a_{21}b_{12} + a_{22}b_{22})c_{22} \\ &= a_{21}b_{11}c_{12} + a_{22}b_{21}c_{12} + a_{21}b_{12}c_{22} + a_{22}b_{22}c_{22} \end{aligned}$$

If we set

$$E = A(BC) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11}c_{11} + b_{12}c_{21} & b_{11}c_{12} + b_{12}c_{22} \\ b_{21}c_{11} + b_{22}c_{21} & b_{21}c_{12} + b_{22}c_{22} \end{pmatrix}$$

then it follows that

$$\begin{aligned} e_{11} &= a_{11}(b_{11}c_{11} + b_{12}c_{21}) + a_{12}(b_{21}c_{11} + b_{22}c_{21}) \\ &= a_{11}b_{11}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{21}c_{11} + a_{12}b_{22}c_{21} \\ e_{12} &= a_{11}(b_{11}c_{12} + b_{12}c_{22}) + a_{12}(b_{21}c_{12} + b_{22}c_{22}) \\ &= a_{11}b_{11}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{21}c_{12} + a_{12}b_{22}c_{22} \\ e_{21} &= a_{21}(b_{11}c_{11} + b_{12}c_{21}) + a_{22}(b_{21}c_{11} + b_{22}c_{21}) \\ &= a_{21}b_{11}c_{11} + a_{21}b_{12}c_{21} + a_{22}b_{21}c_{11} + a_{22}b_{22}c_{21} \\ e_{22} &= a_{21}(b_{11}c_{12} + b_{12}c_{22}) + a_{22}(b_{21}c_{12} + b_{22}c_{22}) \\ &= a_{21}b_{11}c_{12} + a_{21}b_{12}c_{22} + a_{22}b_{21}c_{12} + a_{22}b_{22}c_{22} \end{aligned}$$

Thus

$$d_{11} = e_{11} \qquad d_{12} = e_{12} \qquad d_{21} = e_{21} \qquad d_{22} = e_{22}$$

and hence

$$(AB)C = D = E = A(BC)$$

7. $A^2 = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$, $A^n = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$.

9.

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $A^4 = O$. If $n > 4$, then

$$A^n = A^{n-4}A^4 = A^{n-4}O = O$$

10. (a) The matrix C is symmetric since

$$C^T = (A + B)^T = A^T + B^T = A + B = C$$

(b) The matrix D is symmetric since

$$D^T = (AA)^T = A^T A^T = A^2 = D$$

(c) The matrix $E = AB$ is not symmetric since

$$E^T = (AB)^T = B^T A^T = BA$$

and in general, $AB \neq BA$.

(d) The matrix F is symmetric since

$$F^T = (ABA)^T = A^T B^T A^T = ABA = F$$

(e) The matrix G is symmetric since

$$G^T = (AB + BA)^T = (AB)^T + (BA)^T = B^T A^T + A^T B^T = BA + AB = G$$

(f) The matrix H is not symmetric since

$$H^T = (AB - BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = BA - AB = -H$$

11. (a) The matrix A is symmetric since

$$A^T = (C + C^T)^T = C^T + (C^T)^T = C^T + C = A$$

(b) The matrix B is not symmetric since

$$B^T = (C - C^T)^T = C^T - (C^T)^T = C^T - C = -B$$

(c) The matrix D is symmetric since

$$A^T = (C^T C)^T = C^T (C^T)^T = C^T C = D$$

(d) The matrix E is symmetric since

$$\begin{aligned} E^T &= (C^T C - CC^T)^T = (C^T C)^T - (CC^T)^T \\ &= C^T (C^T)^T - (C^T)^T C^T = C^T C - CC^T = E \end{aligned}$$

(e) The matrix F is symmetric since

$$F^T = ((I + C)(I + C^T))^T = (I + C^T)^T (I + C)^T = (I + C)(I + C^T) = F$$

(e) The matrix G is not symmetric.

$$\begin{aligned} F &= (I + C)(I - C^T) = I + C - C^T - CC^T \\ F^T &= ((I + C)(I - C^T))^T = (I - C^T)^T (I + C)^T \\ &= (I - C)(I + C^T) = I - C + C^T - CC^T \end{aligned}$$

F and F^T are not the same. The two middle terms $C - C^T$ and $-C + C^T$ do not agree.

12. If $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$, then

$$\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{a_{11}a_{22} - a_{12}a_{21}}{d} & 0 \\ 0 & \frac{a_{11}a_{22} - a_{12}a_{21}}{d} \end{pmatrix} = I$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\left[\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\right] = \begin{pmatrix} \frac{a_{11}a_{22} - a_{12}a_{21}}{d} & 0 \\ 0 & \frac{a_{11}a_{22} - a_{12}a_{21}}{d} \end{pmatrix} = I$$

Therefore

$$\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = A^{-1}$$

13. (a) $\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$

(b) $\begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

14. If A were nonsingular and $AB = A$, then it would follow that $A^{-1}AB = A^{-1}A$ and hence that $B = I$. So if $B \neq I$, then A must be singular.

15. Since

$$A^{-1}A = AA^{-1} = I$$

it follows from the definition that A^{-1} is nonsingular and its inverse is A.

16. Since

$$\begin{aligned} A^T(A^{-1})^T &= (A^{-1}A)^T = I \\ (A^{-1})^T A^T &= (AA^{-1})^T = I \end{aligned}$$

it follows that

$$(A^{-1})^T = (A^T)^{-1}$$

17. If $A\mathbf{x} = A\mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, then A must be singular, for if A were nonsingular, then we could multiply by A^{-1} and get

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}A\mathbf{y} \\ \mathbf{x} &= \mathbf{y} \end{aligned}$$

18. For $m = 1$,

$$(A^1)^{-1} = A^{-1} = (A^{-1})^1$$

Assume the result holds in the case $m = k$, that is,

$$(A^k)^{-1} = (A^{-1})^k$$

It follows that

$$(A^{-1})^{k+1}A^{k+1} = A^{-1}(A^{-1})^k A^k A = A^{-1}A = I$$

and

$$A^{k+1}(A^{-1})^{k+1} = AA^k(A^{-1})^k A^{-1} = AA^{-1} = I$$

Therefore

$$(A^{-1})^{k+1} = (A^{k+1})^{-1}$$

and the result follows by mathematical induction.

19. If $A^2 = O$, then

$$(I + A)(I - A) = I + A - A + A^2 = I$$

and

$$(I - A)(I + A) = I - A + A + A^2 = I$$

Therefore $I - A$ is nonsingular and $(I - A)^{-1} = I + A$.

20. If $A^{k+1} = O$, then

$$\begin{aligned} (I + A + \cdots + A^k)(I - A) &= (I + A + \cdots + A^k) - (A + A^2 + \cdots + A^{k+1}) \\ &= I - A^{k+1} = I \end{aligned}$$

and

$$\begin{aligned} (I - A)(I + A + \cdots + A^k) &= (I + A + \cdots + A^k) - (A + A^2 + \cdots + A^{k+1}) \\ &= I - A^{k+1} = I \end{aligned}$$

Therefore $I - A$ is nonsingular and $(I - A)^{-1} = I + A + A^2 + \cdots + A^k$.

21. Since

$$R^TR = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$RR^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

it follows that R is nonsingular and $R^{-1} = R^T$

22.

$$G^2 = \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = I$$

23.

$$\begin{aligned} H^2 = (I - 2\mathbf{u}\mathbf{u}^T)^2 &= I - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T \\ &= I - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}(\mathbf{u}^T\mathbf{u})\mathbf{u}^T \\ &= I - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T = I \quad (\text{since } \mathbf{u}^T\mathbf{u} = 1) \end{aligned}$$

24. In each case, if you square the given matrix, you will end up with the same matrix.

25. (a) If $A^2 = A$, then

$$(I - A)^2 = I - 2A + A^2 = I - 2A + A = I - A$$

(b) If $A^2 = A$, then

$$(I - \frac{1}{2}A)(I + A) = I - \frac{1}{2}A + A - \frac{1}{2}A^2 = I - \frac{1}{2}A + A - \frac{1}{2}A = I$$

and

$$(I + A)(I - \frac{1}{2}A) = I + A - \frac{1}{2}A - \frac{1}{2}A^2 = I + A - \frac{1}{2}A - \frac{1}{2}A = I$$

Therefore $I + A$ is nonsingular and $(I + A)^{-1} = I - \frac{1}{2}A$.

26. (a)

$$D^2 = \begin{pmatrix} d_{11}^2 & 0 & \cdots & 0 \\ 0 & d_{22}^2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & d_{nn}^2 \end{pmatrix}$$

Since each diagonal entry of D is equal to either 0 or 1, it follows that $d_{jj}^2 = d_{jj}$, for $j = 1, \ldots, n$ and hence $D^2 = D$.

(b) If $A = XDX^{-1}$, then

$$A^2 = (XDX^{-1})(XDX^{-1}) = XD(X^{-1}X)DX^{-1} = XDX^{-1} = A$$

27. If A is an involution, then $A^2 = I$ and it follows that

$$B^2 = \frac{1}{4}(I+A)^2 = \frac{1}{4}(I+2A+A^2) = \frac{1}{4}(2I+2A) = \frac{1}{2}(I+A) = B$$
$$C^2 = \frac{1}{4}(I-A)^2 = \frac{1}{4}(I-2A+A^2) = \frac{1}{4}(2I-2A) = \frac{1}{2}(I-A) = C$$

So B and C are both idempotent.

$$BC = \frac{1}{4}(I+A)(I-A) = \frac{1}{4}(I+A-A-A^2) = \frac{1}{4}(I+A-A-I) = O$$

28. $(A^TA)^T = A^T(A^T)^T = A^TA$
$(AA^T)^T = (A^T)^TA^T = AA^T$

29. Let A and B be symmetric $n \times n$ matrices. If $(AB)^T = AB$, then

$$BA = B^TA^T = (AB)^T = AB$$

Conversely, if $BA = AB$, then

$$(AB)^T = B^TA^T = BA = AB$$

30. (a)

$$B^T = (A+A^T)^T = A^T + (A^T)^T = A^T + A = B$$
$$C^T = (A-A^T)^T = A^T - (A^T)^T = A^T - A = -C$$

(b) $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$

34. False. For example, if

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

then

$$A\mathbf{x} = B\mathbf{x} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

however, $A \neq B$.

35. False. For example, if

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then it is easy to see that both A and B must be singular, however, $A + B = I$, which is nonsingular.

36. True. If A and B are nonsingular, then their product AB must also be nonsingular. Using the result from Exercise 23, we have that $(AB)^T$ is nonsingular and $((AB)^T)^{-1} = ((AB)^{-1})^T$. It follows then that

$$((AB)^T)^{-1} = ((AB)^{-1})^T = (B^{-1}A^{-1})^T = (A^{-1})^T(B^{-1})^T$$

5 ELEMENTARY MATRICES

2. (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, type I

(b) The given matrix is not an elementary matrix. Its inverse is given by

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$, type III

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, type II

3. (a) $\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

4. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

5. (a) $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

(b) $F = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) Since

$$C = FB = FEA$$

where F and E are elementary matrices, it follows that C is row equivalent to A.

6. (a) $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

The product $U = E_3E_2E_1A$ is upper triangular.

$$U = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 3 & 3 \\ 0 & 0 & 7 \end{pmatrix}$$

(b) $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$, $E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

The product $L = E_1^{-1}E_2^{-1}E_3^{-1}$ is lower triangular.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

7. A can be reduced to the identity matrix using three row operations

$$\begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The elementary matrices corresponding to the three row operations are

$$E_1 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

So

$$E_3E_2E_1A = I$$

and hence

$$A = E_1^{-1}E_3^{-1}E_3^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

and $A^{-1} = E_3E_2E_1$.

8. (b) $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

9. (a) $\begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

10. (e) $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

11. $A^{-1} = \begin{pmatrix} \frac{4}{5} & -\frac{7}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix}$

(a) $X = A^{-1}B = \begin{pmatrix} \frac{1}{5} & 4 \\ \frac{1}{5} & -1 \end{pmatrix}$

(b) $Y = BA^{-1} = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{4}{5} & -\frac{7}{5} \end{pmatrix}$

12. (b) $XA + B = C$
$X = (C - B)A^{-1}$
$= \begin{pmatrix} 8 & -14 \\ -13 & 19 \end{pmatrix}$

(d) $XA + C = X$
$XA - XI = -C$
$X(A - I) = -C$
$X = -C(A - I)^{-1}$
$= \begin{pmatrix} 2 & -4 \\ -3 & 6 \end{pmatrix}$

13. (a) If E is an elementary matrix of type I or type II, then E is symmetric. Thus $E^T = E$ is an elementary matrix of the same type. If E is the elementary matrix of type III formed by adding α times the ith row of the identity matrix to the jth row, then E^T is the elementary matrix of type III formed from the identity matrix by adding α times the jth row to the ith row.

(b) In general, the product of two elementary matrices will not be an elementary matrix. Generally, the product of two elementary matrices will be a matrix formed from the identity matrix by the performance of two row operations. For example, if

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

then E_1 and E_2 are elementary matrices, but

$$E_1E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

is not an elementary matrix.

14. If $T = UR$, then

$$t_{ij} = \sum_{k=1}^{n} u_{ik}r_{kj}$$

Since U and R are upper triangular

$$\begin{aligned} u_{i1} &= u_{i2} = \cdots = u_{i,i-1} = 0 \\ r_{j+1,j} &= r_{j+2,j} = \cdots - r_{nj} = 0 \end{aligned}$$

If $i > j$, then

$$\begin{aligned} t_{ij} &= \sum_{k=1}^{j} u_{ik}r_{kj} + \sum_{k=j+1}^{n} u_{ik}r_{kj} \\ &= \sum_{k=1}^{j} 0\, r_{kj} + \sum_{k=j+1}^{n} u_{ik}0 \\ &= 0 \end{aligned}$$

Therefore T is upper triangular.

If $i = j$, then

$$\begin{aligned} t_{jj} = t_{ij} &= \sum_{k=1}^{i-1} u_{ik}r_{kj} + u_{jj}r_{jj} + \sum_{k=j+1}^{n} u_{ik}r_{kj} \\ &= \sum_{k=1}^{i-1} 0\, r_{kj} + u_{jj}r_{jj} + \sum_{k=j+1}^{n} u_{ik}0 \\ &= u_{jj}r_{jj} \end{aligned}$$

Therefore

$$t_{jj} = u_{jj}r_{jj} \qquad j = 1, \ldots, n$$

15. If we set $\mathbf{x} = (2, 1 - 4)^T$, then

$$A\mathbf{x} = 2\mathbf{a}_1 + 1\mathbf{a}_2 - 4\mathbf{a}_3 = \mathbf{0}$$

Thus $\mathbf{x}$ is a nonzero solution to the system $A\mathbf{x} = \mathbf{0}$. But if a homogeneous system has a nonzero solution, then it must have infinitely many solutions. In particular, if c is any scalar, then $c\mathbf{x}$ is also a solution to the system since

$$A(c\mathbf{x}) = cA\mathbf{x} = c\mathbf{0} = \mathbf{0}$$

Since $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, it follows that the matrix A must be singular. (See Theorem 1.5.2)

16. If $\mathbf{a}_1 = 3\mathbf{a}_2 - 2\mathbf{a}_3$, then

$$\mathbf{a}_1 - 3\mathbf{a}_2 + 2\mathbf{a}_3 = \mathbf{0}$$

Therefore $\mathbf{x} = (1, -3, 2)^T$ is a nontrivial solution to $A\mathbf{x} = \mathbf{0}$. It follows from Theorem 1.5.2 that A must be singular.

17. If $\mathbf{x}_0 \neq \mathbf{0}$ and $A\mathbf{x}_0 = B\mathbf{x}_0$, then $C\mathbf{x}_0 = \mathbf{0}$ and it follows from Theorem 1.5.2 that C must be singular.

18. If B is singular, then it follows from Theorem 1.5.2 that there exists a nonzero vector $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$. If $C = AB$, then

$$C\mathbf{x} = AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$$

Thus, by Theorem 1.5.2, C must also be singular.

19. (a) If U is upper triangular with nonzero diagonal entries, then using row operation II, U can be transformed into an upper triangular matrix with 1's on the diagonal. Row operation III can then be used to eliminate all of the entries above the diagonal. Thus, U is row equivalent to I and hence is nonsingular.

(b) The same row operations that were used to reduce U to the identity matrix will transform I into U^{-1}. Row operation II applied to I will just change the values of the diagonal entries. When the row operation III steps referred to in part (a) are applied to a diagonal matrix, the entries above the diagonal are filled in. The resulting matrix, U^{-1}, will be upper triangular.

20. Since A is nonsingular it is row equivalent to I. Hence, there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_1 A = I$$

It follows that

$$A^{-1} = E_k \cdots E_1$$

and

$$E_k \cdots E_1 B = A^{-1}B = C$$

The same row operations that reduce A to I, will transform B to C. Therefore, the reduced row echelon form of $(A \mid B)$ will be $(I \mid C)$.

21. (a) If the diagonal entries of D_1 are $\alpha_1, \alpha_2, \ldots, \alpha_n$ and the diagonal entries of D_2 are $\beta_1, \beta_2, \ldots, \beta_n$, then D_1D_2 will be a diagonal matrix with diagonal entries $\alpha_1\beta_1, \ldots, \alpha_n\beta_n$ and D_2D_1 will be a diagonal matrix with diagonal entries $\beta_1\alpha_1, \beta_2\alpha_2, \ldots, \beta_n\alpha_n$. Since the two have the same diagonal entries, it follows that $D_1D_2 = D_2D_1$.

(b)

$$\begin{aligned} AB &= A(a_0I + a_1A + \cdots + a_kA^k) \\ &= a_0A + a_1A^2 + \cdots + a_kA^{k+1} \\ &= (a_0I + a_1A + \cdots + a_kA^k)A \\ &= BA \end{aligned}$$

22. If A is symmetric and nonsingular, then

$$(A^{-1})^T = (A^{-1})^T(AA^{-1}) = ((A^{-1})^T A^T)A^{-1} = A^{-1}$$

23. If A is row equivalent to B, then there exist elementary matrices $E_1, E_2, \dots, E_k$ such that

$$A = E_k E_{k-1} \cdots E_1 B$$

Each of the E_i's is invertible and E_i^{-1} is also an elementary matrix (Theorem 1.4.1). Thus

$$B = E_1^{-1} E_2^{-1} \cdots E_k^{-1} A$$

and hence B is row equivalent to A.

24. (a) If A is row equivalent to B, then there exist elementary matrices $E_1, E_2, \dots, E_k$ such that

$$A = E_k E_{k-1} \cdots E_1 B$$

Since B is row equivalent to C, there exist elementary matrices $H_1, H_2, \dots, H_j$ such that

$$B = H_j H_{j-1} \cdots H_1 C$$

Thus

$$A = E_k E_{k-1} \cdots E_1 H_j H_{j-1} \cdots H_1 C$$

and hence A is row equivalent to C.

(b) If A and B are nonsingular $n \times n$ matrices, then A and B are row equivalent to I. Since A is row equivalent to I and I is row equivalent to B, it follows from part (a) that A is row equivalent to B.

25. If U is any row echelon form of A, then A can be reduced to U using row operations, so A is row equivalent to U. If B is row equivalent to A, then it follows from the result in Exercise 24(a) that B is row equivalent to U.

26. If B is row equivalent to A, then there exist elementary matrices $E_1, E_2, \dots, E_k$ such that

$$B = E_k E_{k-1} \cdots E_1 A$$

Let $M = E_k E_{k-1} \cdots E_1$. The matrix M is nonsingular since each of the E_i's is nonsingular.

Conversely, suppose there exists a nonsingular matrix M such that $B = MA$. Since M is nonsingular, it is row equivalent to I. Thus, there exist elementary matrices $E_1, E_2, \dots, E_k$ such that

$$M = E_k E_{k-1} \cdots E_1 I$$

It follows that

$$B = MA = E_k E_{k-1} \cdots E_1 A$$

Therefore, B is row equivalent to A.

27. If A is nonsingular, then A is row equivalent to I. If B is row equivalent to A, then using the result from Exercise 24(a), we can conclude that B is row equivalent to I. Therefore, B must be nonsingular. So it is not possible for B to be singular and also be row equivalent to a nonsingular matrix.

28. (a) The system $V\mathbf{c} = \mathbf{y}$ is given by

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & & & & \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{pmatrix}$$

Comparing the ith row of each side, we have

$$c_1 + c_2 x_i + \cdots + c_{n+1} x_i^n = y_i$$

Thus

$$p(x_i) = y_i \qquad i = 1, 2, \ldots, n+1$$

(b) If $x_1, x_2, \ldots, x_{n+1}$ are distinct and $V\mathbf{c} = \mathbf{0}$, then we can apply part (a) with $\mathbf{y} = \mathbf{0}$. Thus if $p(x) = c_1 + c_2x + \cdots + c_{n+1}x^n$, then

$$p(x_i) = 0 \qquad i = 1, 2, \ldots, n+1$$

The polynomial $p(x)$ has $n+1$ roots. Since the degree of $p(x)$ is less than $n+1$, $p(x)$ must be the zero polynomial. Hence

$$c_1 = c_2 = \cdots = c_{n+1} = 0$$

Since the system $V\mathbf{c} = \mathbf{0}$ has only the trivial solution, the matrix V must be nonsingular.

29. True. If A is row equivalent to I, then A is nonsingular, so if $AB = AC$, then we can multiply both sides of this equation by A^{-1}.

$$\begin{aligned} A^{-1}AB &= A^{-1}AC \\ B &= C \end{aligned}$$

30. True. If E and F are elementary matrices, then they are both nonsingular and the product of two nonsingular matrices is a nonsingular matrix. Indeed, $G^{-1} = F^{-1}E^{-1}$.

31. True. If $\mathbf{a} + \mathbf{a}_2 = \mathbf{a}_3 + 2\mathbf{a}_4$, then

$$\mathbf{a} + \mathbf{a}_2 - \mathbf{a}_3 - 2\mathbf{a}_4 = \mathbf{0}$$

If we let $\mathbf{x} = (1, 1, -1, -2)^T$, then $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{0}$. Since $\mathbf{x} \neq \mathbf{0}$ the matrix A must be singular.

32. False. Let I be the 2×2 identity matrix and let $A = I$, $B = -I$, and

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Since B and C are nonsingular, they are both row equivalent to A; however,

$$B + C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is singular, so it cannot be row equivalent to A.

6 PARTITIONED MATRICES

2. $B = A^TA = \begin{pmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{pmatrix} (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) = \begin{pmatrix} \mathbf{a}_1^T\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{a}_2 & \cdots & \mathbf{a}_1^T\mathbf{a}_n \\ \mathbf{a}_2^T\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{a}_2 & \cdots & \mathbf{a}_2^T\mathbf{a}_n \\ \vdots & & & \\ \mathbf{a}_n^T\mathbf{a}_1 & \mathbf{a}_n^T\mathbf{a}_2 & \cdots & \mathbf{a}_n^T\mathbf{a}_n \end{pmatrix}$

3. (a) $A\mathbf{b_1} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$, $A\mathbf{b_2} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

(b) $\vec{\mathbf{a_1}}B = \begin{pmatrix} 5 & 3 \end{pmatrix}$, $\vec{\mathbf{a_2}}B = \begin{pmatrix} -5 & 4 \end{pmatrix}$

(c) $AB = \begin{pmatrix} 5 & 3 \\ -5 & 4 \end{pmatrix}$

4. (a) $$\begin{pmatrix} O & I \\ I & O \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} B_{21} & B_{22} \\ B_{11} & B_{12} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 3 \\ \hline 1 & 2 & 3 & 1 \\ 1 & 2 & 0 & 0 \end{array}\right)$$

(b) $$\begin{pmatrix} C & O \\ O & C \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} CB_{11} & CB_{12} \\ CB_{21} & CB_{22} \end{pmatrix} = \left(\begin{array}{rr|rr} 0 & 0 & 3 & 1 \\ 0 & 0 & -3 & -1 \\ \hline -1 & -1 & -1 & -2 \\ 1 & 1 & 1 & 2 \end{array}\right)$$

(c) $$\begin{pmatrix} D & O \\ O & I \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} DB_{11} & DB_{12} \\ B_{21} & B_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} 3 & 6 & 9 & 3 \\ 3 & 6 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 3 \end{array}\right)$$

(d) $$\begin{pmatrix} E & O \\ O & E \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} EB_{11} & EB_{12} \\ EB_{21} & EB_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 1 \\ \hline 2 & 1 & 1 & 3 \\ 1 & 0 & 0 & 1 \end{array}\right)$$

5. (a) $$\begin{pmatrix} 2 & -1 & 3 \\ 4 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 1 \\ 4 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 13 & 3 & 12 \\ 4 & 7 & 19 \end{pmatrix}$$

(b) $$\left(\begin{array}{cc} 1 & 2 \\ 2 & 1 \\ 4 & 0 \\ \hline 1 & 0 \end{array}\right)\left(\begin{array}{rrr|r} 2 & -1 & 3 & 1 \\ 4 & -1 & 0 & 2 \end{array}\right) = \left(\begin{array}{rrr|r} 10 & -3 & 3 & 5 \\ 8 & -3 & 6 & 4 \\ 8 & -4 & 12 & 4 \\ \hline 2 & -1 & 3 & 1 \end{array}\right)$$

(c) Let

$$A_{11} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{pmatrix} \quad A_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad A_{22} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

The block multiplication is performed as follows:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{pmatrix} = \begin{pmatrix} A_{11}A_{11}^T + A_{12}A_{12}^T & A_{11}A_{21}^T + A_{12}A_{22}^T \\ A_{21}A_{11}^T + A_{22}A_{12}^T & A_{21}A_{21}^T + A_{22}A_{22}^T \end{pmatrix}$$

$$= \left(\begin{array}{cc|c} \frac{5}{8} & 0 & 0 \\ 0 & \frac{5}{8} & 0 \\ \hline 0 & 0 & 2 \end{array}\right)$$

(d) $$\left(\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \left(\begin{array}{cc} 1 & 0 \\ 2 & -9 \\ 3 & -8 \\ \hline 4 & -7 \\ 5 & -6 \end{array}\right) = \left(\begin{array}{cc} 3 & -8 \\ 2 & -9 \\ 1 & 0 \\ \hline 5 & -6 \\ 4 & -7 \end{array}\right)$$

6. (a)

$$\begin{aligned} XY^T &= \mathbf{x}_1\mathbf{y}_1^T + \mathbf{x}_2\mathbf{y}_2^T + \mathbf{x}_3\mathbf{y}_3^T \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 16 \\ 1 & 4 \end{pmatrix} \end{aligned}$$

(b) Since $\mathbf{y}_j\mathbf{x}_j^T = (\mathbf{x}_j\mathbf{y}_j^T)^T$ for $j = 1,\ 2,\ 3$, the outer product expansion of YX^T is just the transpose of the outer product expansion of XY^T. Thus,

$$\begin{aligned} YX^T &= \mathbf{y}_1\mathbf{x}_1^T + \mathbf{y}_2\mathbf{x}_2^T + \mathbf{y}_3\mathbf{x}_3^T \\ &= \begin{pmatrix} 2 & 1 \\ 8 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ 16 & 4 \end{pmatrix} \end{aligned}$$

7. It is possible to perform both block multiplications. To see this, suppose A_{11} is a $k \times r$ matrix, A_{12} is a $k \times (n-r)$ matrix, A_{21} is an $(m-k) \times r$ matrix and A_{22} is $(m-k) \times (n-r)$. It is possible to perform the block multiplication of AA^T since the matrix multiplications $A_{11}A_{11}^T$, $A_{11}A_{21}^T$, $A_{12}A_{12}^T$, $A_{12}A_{22}^T$, $A_{21}A_{11}^T$, $A_{21}A_{21}^T$, $A_{22}A_{12}^T$, $A_{22}A_{22}^T$ are all possible. It is possible to perform the block multiplication of A^TA since the matrix multiplications $A_{11}^TA_{11}$, $A_{11}^TA_{12}$, $A_{21}^TA_{21}$, $A_{21}^TA_{11}$, $A_{12}^TA_{12}$, $A_{22}^TA_{21}$, $A_{22}^TA_{22}$ are all possible.

8. $AX = A(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r) = (A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_r)$
$B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r)$
$AX = B$ if and only if the column vectors of AX and B are equal

$$A\mathbf{x}_j = \mathbf{b}_j \qquad j = 1, \ldots, r$$

9. (a) Since D is a diagonal matrix, its jth column will have d_{jj} in the jth row and the other entries will all be 0. Thus $\mathbf{d}_j = d_{jj}\mathbf{e}_j$ for $j = 1, \ldots, n$.

(b)

$$\begin{aligned} AD &= A(d_{11}\mathbf{e}_1,\ d_{22}\mathbf{e}_2, \ldots, d_{nn}\mathbf{e}_n) \\ &= (d_{11}A\mathbf{e}_1,\ d_{22}A\mathbf{e}_2, \ldots, d_{nn}A\mathbf{e}_n) \\ &= (d_{11}\mathbf{a}_1,\ d_{22}\mathbf{a}_2, \ldots, d_{nn}\mathbf{a}_n) \end{aligned}$$

10. (a)

$$U\Sigma = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 \\ O \end{pmatrix} = U_1\Sigma_1 + U_2 O = U_1\Sigma_1$$

(b) If we let $X = U\Sigma$, then

$$X = U_1\Sigma_1 = (\sigma_1\mathbf{u}_1, \sigma_2\mathbf{u}_2, \ldots, \sigma_n\mathbf{u}_n)$$

and it follows that

$$A = U\Sigma V^T = XV^T = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^T + \cdots + \sigma_n\mathbf{u}_n\mathbf{v}_n^T$$

11.

$$\begin{pmatrix} A_{11}^{-1} & C \\ O & A_{22}^{-1} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix} = \begin{pmatrix} I & A_{11}^{-1}A_{12} + CA_{22} \\ O & I \end{pmatrix}$$

If

$$A_{11}^{-1}A_{12} + CA_{22} = O$$

then

$$C = -A_{11}^{-1}A_{12}A_{22}^{-1}$$

Let

$$B = \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ O & A_{22}^{-1} \end{pmatrix}$$

Since $AB = BA = I$, it follows that $B = A^{-1}$.

12. Let $\mathbf{0}$ denote the zero vector in $\mathbb{R}^n$. If A is singular, then there exists a vector $\mathbf{x}_1 \neq \mathbf{0}$ such that $A\mathbf{x}_1 = \mathbf{0}$. If we set

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{pmatrix}$$

then

$$M\mathbf{x} = \begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} A\mathbf{x}_1 + O\mathbf{0} \\ O\mathbf{x}_1 + B\mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

By Theorem 1.5.2, M must be singular. Similarly, if B is singular, then there exists a vector $\mathbf{x}_2 \neq \mathbf{0}$ such that $B\mathbf{x}_2 = \mathbf{0}$. So if we set

$$\mathbf{x} = \begin{pmatrix} \mathbf{0} \\ \mathbf{x}_2 \end{pmatrix}$$

then $\mathbf{x}$ is a nonzero vector and $M\mathbf{x}$ is equal to the zero vector.

15.

$$A^{-1} = \begin{pmatrix} O & I \\ I & -B \end{pmatrix}, \quad A^2 = \begin{pmatrix} I & B \\ B & I \end{pmatrix}, \quad A^3 = \begin{pmatrix} B & I \\ I & 2B \end{pmatrix}$$

and hence

$$A^{-1} + A^2 + A^3 = \begin{pmatrix} I+B & 2I+B \\ 2I+B & I+B \end{pmatrix}$$

16. The block form of S^{-1} is given by

$$S^{-1} = \begin{pmatrix} I & -A \\ O & I \end{pmatrix}$$

It follows that

$$\begin{aligned} S^{-1}MS &= \begin{pmatrix} I & -A \\ O & I \end{pmatrix} \begin{pmatrix} AB & O \\ B & O \end{pmatrix} \begin{pmatrix} I & A \\ O & I \end{pmatrix} \\ &= \begin{pmatrix} I & -A \\ O & I \end{pmatrix} \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix} \\ &= \begin{pmatrix} O & O \\ B & BA \end{pmatrix} \end{aligned}$$

17. The block multiplication of the two factors yields

$$\begin{pmatrix} I & O \\ B & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ O & C \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ BA_{11} & BA_{12} + C \end{pmatrix}$$

If we equate this matrix with the block form of A and solve for B and C, we get

$$B = A_{21}A_{11}^{-1} \qquad \text{and} \qquad C = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

To check that this works note that

$$\begin{aligned} BA_{11} &= A_{21}A_{11}^{-1}A_{11} = A_{21} \\ BA_{12} + C &= A_{21}A_{11}^{-1}A_{12} + A_{22} - A_{21}A_{11}^{-1}A_{12} = A_{22} \end{aligned}$$

and hence

$$\begin{pmatrix} I & O \\ B & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ O & C \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = A$$

18. In order for the block multiplication to work, we must have

$$XB = S \qquad \text{and} \qquad YM = T$$

Since both B and M are nonsingular, we can satisfy these conditions by choosing $X = SB^{-1}$ and $Y = TM^{-1}$.

19. (a)

$$BC = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} (c) = \begin{pmatrix} b_1 c \\ b_2 c \\ \vdots \\ b_n c \end{pmatrix} = c\mathbf{b}$$

(b)

$$\begin{aligned} A\mathbf{x} &= (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= \mathbf{a}_1(x_1) + \mathbf{a}_2(x_2) + \cdots + \mathbf{a}_n(x_n) \end{aligned}$$

(c) It follows from parts (a) and (b) that

$$\begin{aligned} A\mathbf{x} &= \mathbf{a}_1(x_1) + \mathbf{a}_2(x_2) + \cdots + \mathbf{a}_n(x_n) \\ &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n \end{aligned}$$

20. If $A\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$, then

$$\mathbf{a}_j = A\mathbf{e}_j = \mathbf{0} \quad \text{for} \quad j = 1, \ldots, n$$

and hence A must be the zero matrix.

21. If

$$B\mathbf{x} = C\mathbf{x} \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n$$

then

$$(B - C)\mathbf{x} = \mathbf{0} \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n$$

It follows from Exercise 20 that

$$\begin{aligned} B - C &= O \\ B &= C \end{aligned}$$

22. (a)

$$\begin{pmatrix} A^{-1} & \mathbf{0} \\ -\mathbf{c}^T A^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & \mathbf{a} \\ \mathbf{c}^T & \beta \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} A^{-1} & \mathbf{0} \\ -\mathbf{c}^T A^{-1} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ b_{n+1} \end{pmatrix}$$

$$\begin{pmatrix} I & A^{-1}\mathbf{a} \\ \mathbf{0}^T & -\mathbf{c}^T A^{-1}\mathbf{a} + \beta \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} A^{-1}\mathbf{b} \\ -\mathbf{c}^T A^{-1}\mathbf{b} + b_{n+1} \end{pmatrix}$$

(b) If

$$\mathbf{y} = A^{-1}\mathbf{a} \qquad \text{and} \qquad \mathbf{z} = A^{-1}\mathbf{b}$$

then

$$(-\mathbf{c}^T\mathbf{y} + \beta)x_{n+1} = -\mathbf{c}^T\mathbf{z} + b_{n+1}$$
$$x_{n+1} = \frac{-\mathbf{c}^T\mathbf{z} + b_{n+1}}{-\mathbf{c}^T\mathbf{y} + \beta} \quad (\beta - \mathbf{c}^T\mathbf{y} \neq 0)$$

and

$$\mathbf{x} + x_{n+1}A^{-1}\mathbf{a} = A^{-1}\mathbf{b}$$
$$\mathbf{x} = A^{-1}\mathbf{b} - x_{n+1}A^{-1}\mathbf{a} = \mathbf{z} - x_{n+1}\mathbf{y}$$

MATLAB EXERCISES

1. In parts (a) and (c), it should turn out that $A1 = A4$ and $A2 = A3$. In part (b) and (d), $A1 = A3$ and $A2 = A4$. Exact equality might not occur in parts (c) and (d) because of roundoff error.

2. The solution $\mathbf{x}$ obtained using the \ operation will be more accurate and yield the smaller residual vector. The computation of $\mathbf{x}$ is also more efficient since the solution is computed using Gaussian elimination with partial pivoting and this involves less arithmetic than computing the inverse matrix and multiplying it times $\mathbf{b}$.

3. (a) Since $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, it follows from Theorem 1.5.2 that A is singular.

(b) The columns of B are all multiples of $\mathbf{x}$. Indeed,

$$B = (\mathbf{x}, 2\mathbf{x}, 3\mathbf{x}, 4\mathbf{x}, 5\mathbf{x}, 6\mathbf{x})$$

and hence

$$AB = (A\mathbf{x}, 2A\mathbf{x}, 3A\mathbf{x}, 4A\mathbf{x}, 5A\mathbf{x}, 6A\mathbf{x}) = O$$

(c) If $D = B + C$, then

$$AD = AB + AC = O + AC = AC$$

4. By construction, B is upper triangular with diagonal entries are all equal to 1. Thus, B is row equivalent to I, and, hence, B is nonsingular. If one changes B by setting $b_{9,1} = -1/128$ and computes $B\mathbf{x}$, the result is the zero vector. Since $\mathbf{x} \neq \mathbf{0}$, the matrix B must be singular.

5. (a) Since A is nonsingular, its reduced row echelon form is I. If $E_1, \ldots, E_k$ are elementary matrices such that $E_k \cdots E_1 A = I$, then these same matrices can be used to transform $(A \;\; \mathbf{b})$ to its reduced row echelon form U. It follows then that

$$U = E_k \cdots E_1(A \;\; \mathbf{b}) = A^{-1}(A \;\; \mathbf{b}) = (I \;\; A^{-1}\mathbf{b})$$

Thus, the last column of U should be equal to the solution $\mathbf{x}$ of the system $A\mathbf{x} = \mathbf{b}$.

(b) After the third column of A is changed, the new matrix A is now singular. Examining the last row of the reduced row echelon form of the augmented matrix $(A \; \mathbf{b})$, we see that the system is inconsistent.

(c) The system $A\mathbf{x} = \mathbf{c}$ is consistent since $\mathbf{y}$ is a solution. There is a free variable x_3, so the system will have infinitely many solutions.

(f) The vector $\mathbf{v}$ is a solution since

$$A\mathbf{v} = A(\mathbf{w} + 3\mathbf{z}) = A\mathbf{w} + 3A\mathbf{z} = \mathbf{c}$$

For this solution, the free variable $x_3 = v_3 = 3$. To determine the general solution just set $\mathbf{x} = \mathbf{w} + t\mathbf{z}$. This will give the solution corresponding to $x_3 = t$ for any real number t.

6. (c) There will be no walks of even length from V_i to V_j whenever $i + j$ is odd.

(d) There will be no walks of length k from V_i to V_j whenever $i + j + k$ is odd.

(e) The conjecture is still valid for the graph containing the additional edges.
(f) If the edge $\{V_6, V_8\}$ is included, then the conjecture is no longer valid. There is now a walk of length 1 from V_6 to V_8 and $i + j + k = 6 + 8 + 1$ is odd.

8. The change in part (b) should not have a significant effect on the survival potential for the turtles. The change in part (c) will effect the $(2, 2)$ and $(3, 2)$ of the Leslie matrix. The new values for these entries will be $l_{22} = 0.9540$ and $l_{32} = 0.0101$. With these values, the Leslie population model should predict that the survival period will double but the turtles will still eventually die out.

9. (b) $\mathbf{x1} = \mathbf{c} - V\mathbf{x2}$.

10. (a)

$$A^{2k} = \begin{pmatrix} I & kB \\ kB & I \end{pmatrix}$$

This can be proved using mathematical induction. In the case $k = 1$

$$A^2 = \begin{pmatrix} O & I \\ I & B \end{pmatrix} \begin{pmatrix} O & I \\ I & B \end{pmatrix} = \begin{pmatrix} I & B \\ B & I \end{pmatrix}$$

If the result holds for $k = m$

$$A^{2m} = \begin{pmatrix} I & mB \\ mB & I \end{pmatrix}$$

then

$$\begin{aligned} A^{2m+2} &= A^2 A^{2m} \\ &= \begin{pmatrix} I & B \\ B & I \end{pmatrix} \begin{pmatrix} I & mB \\ mB & I \end{pmatrix} \\ &= \begin{pmatrix} I & (m+1)B \\ (m+1)B & I \end{pmatrix} \end{aligned}$$

It follows by mathematical induction that the result holds for all positive integers k.

(b)

$$A^{2k+1} = AA^{2k} = \begin{pmatrix} O & I \\ I & B \end{pmatrix} \begin{pmatrix} I & kB \\ kB & I \end{pmatrix} = \begin{pmatrix} kB & I \\ I & (k+1)B \end{pmatrix}$$

11. (a) By construction, the entries of A were rounded to the nearest integer. The matrix $B = A^T A$ must also have integer entries and it is symmetric since

$$B^T = (A^T A)^T = A^T (A^T)^T = A^T A = B$$