CHAPTER ONE

Student's Solutions Manual

Anna Amirdjanova
University of Michigan, Ann Arbor

Neil A. Weiss
Arizona State University

FOR

A Course In

Probability

Neil A. Weiss
Arizona State University

PEARSON
Addison Wesley

Boston San Francisco New York
London Toronto Sydney Tokyo Singapore Madrid
Mexico City Munich Paris Cape Town Hong Kong Montreal

Publisher: Greg Tobin
Editor-in-Chief: Deirdre Lynch
Associate Editor: Sara Oliver Gordus
Editorial Assistant: Christina Lepre
Production Coordinator: Kayla Smith-Tarbox
Senior Author Support/Technology Specialist: Joe Vetere
Compositor: Carol A. Weiss
Accuracy Checker: Delray Schultz
Proofreader: Carol A. Weiss

Preface

This *Student's Solutions Manual* (SSM) is designed to be used with the book *A Course in Probability* (CIP) by Neil A. Weiss. The SSM provides complete and detailed solutions to every fourth end-of-section exercise and to all review exercises. Thus, for end-of-section exercises, you will find solutions to problems numbered 1, 5, 9, . . . , whereas, solutions to all review problems are presented.

The solutions in this SSM employ precisely the same notation, format, and style as those given to the examples in CIP. Consequently, you need only concentrate on the exercise solutions themselves without having to struggle with new notations or conventions.

We would like to express our appreciation to all the people at Addison-Wesley who helped make this SSM possible. In particular, we thank Deirdre Lynch, Sara Oliver Gordus, Christina Lepre, Kayla Smith-Tarbox, and Joe Vetere. And, in addition, we thank Carol Weiss for her outstanding job of composition and proofreading.

Ann Arbor, Michigan *A.A.*
Prescott, Arizona *N.A.W.*

Chapter 1

Probability Basics

1.1 From Percentages to Probabilities

Basic Exercises

1.1
a) The population under consideration consists of the five senators specified in the problem statement, namely, Graham, Baucus, Conrad, Murkowski, and Kyl.
b) Because three of the five senators are Democrats and each senator is equally likely to be the one selected, the probability is $3/5$, or 0.6, that the chosen senator is a Democrat.

1.5 From the table, we see that the total number of murder cases during the year in question in which the person murdered was between 20 and 59 years old, inclusive, is

$$2{,}916 + 2{,}175 + \cdots + 372 = 11{,}527.$$

a) The number of these murder cases in which the person murdered was between 40 and 44 years old, inclusive, is 1213. Hence, the probability that the murder victim of the case selected was between 40 and 44 years old, inclusive, is $1{,}213/11{,}527 \approx 0.105$.
b) We see that the number of these murder cases in which the person murdered was 25 years old or older is $11{,}527 - 2{,}916 = 8{,}611$. Consequently, the probability that the murder victim of the case selected was 25 years old or older is $8{,}611/11{,}527 \approx 0.747$.
c) The number of these murder cases in which the person murdered was between 45 and 59 years old, inclusive, is

$$888 + 540 + 372 = 1800.$$

Hence, the probability that the murder victim of the case selected was between 45 and 59 years old, inclusive, is $1{,}800/11{,}527 \approx 0.156$.
d) The number of these murder cases in which the person murdered was either under 30 or over 54 is

$$2916 + 2175 + 372 = 5463.$$

Consequently, the probability that the murder victim of the case selected was either under 30 or over 54 is $5{,}463/11{,}527 \approx 0.474$.
e) The population under consideration consists of all murder cases during the year in question in which the person murdered was between 20 and 59 years old, inclusive.

1.9 Note that 2004 was a leap year, so it had 366 days. Now, consider the random experiment of selecting one 2004 U.S. state governor at random, and let E denote the event that the governor chosen is a Republican. We know that $P(E) = 28/50 = 0.56$. Thus, from the frequentist interpretation of probability, if we independently repeat the random experiment n times, then the proportion of times that event E occurs will approximately equal 0.56. Consequently, in 366 repetitions of the random experiment, we would expect event E to occur roughly $0.56 \cdot 366 = 204.96$ times. In other words, if on each of the 366 days of 2004, one U.S. state governor was randomly selected to read the invocation on a popular radio program, then we would expect a Republican to be chosen on approximately 205 of those days.

Advanced Exercises

1.13
a) As the odds against Fusaichi Pegasus were 3 to 5, the probability that Fusaichi Pegasus would win the race is $5/(5+3) = 0.625$.
b) Because the odds against Red Bullet were 9 to 2, the probability that Red Bullet would win the race is $2/(2+9) \approx 0.182$.

1.2 Set Theory

Note: For convenience, we use "iff" to represent the phrase "if and only if."

Basic Exercises

1.17 Answers will vary, but the subsets of $\mathcal{R}^2$ portrayed in the solution to Exercise 1.16 provide one possibility. For another possibility, let $U = \{a, b, c, d, e, f\}$ and consider the following four subsets of U: $\{a, b, e\}$, $\{b, c, f\}$, $\{c, d, e\}$, $\{a, d, f\}$. We note that each pairwise intersection is nonempty and, hence, in particular, the four sets are not pairwise disjoint; however, every three sets have an empty intersection.

1.21 Refer to Definition 1.5 on page 16.
a) We have

$$\begin{aligned}\{0,1\}^3 &= \{0,1\} \times \{0,1\} \times \{0,1\} \\ &= \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}.\end{aligned}$$

Hence, the members of $\{0,1\}^3$ are the eight ordered triplets inside the curly braces on the second line of the preceding display.
b) We have

$$\{0,1\} \times \{0,1\} \times \{1,2\} = \{(0,0,1), (0,0,2), (0,1,1), (0,1,2), (1,0,1), (1,0,2), (1,1,1), (1,1,2)\}.$$

Thus, the members of $\{0,1\} \times \{0,1\} \times \{1,2\}$ are the eight ordered triplets inside the curly braces on the right of the preceding display.
c) We have

$$\begin{aligned}\big(\{a,b\} \cup \{c,d,e\}\big) \times \{f,g,h\} &= \{a,b,c,d,e\} \times \{f,g,h\} \\ &= \{(a,f), (a,g), (a,h), (b,f), (b,g), (b,h), (c,f), (c,g), \\ &\quad\ (c,h), (d,f), (d,g), (d,h), (e,f), (e,g), (e,h)\}.\end{aligned}$$

Hence, the members of $\big(\{a,b\} \cup \{c,d,e\}\big) \times \{f,g,h\}$ are the 15 ordered pairs shown in the preceding display.
d) We have

$$\{a,b\} \times \{f,g,h\} = \{(a,f), (a,g), (a,h), (b,f), (b,g), (b,h)\}$$

and

$$\{c,d,e\} \times \{f,g,h\} = \{(c,f), (c,g), (c,h), (d,f), (d,g), (d,h), (e,f), (e,g), (e,h)\}.$$

Therefore,

$$\big(\{a,b\}\times\{f,g,h\}\big)\cup\big(\{c,d,e\}\times\{f,g,h\}\big)$$
$$=\{(a,f),(a,g),(a,h),(b,f),(b,g),(b,h),(c,f),(c,g),(c,h),(d,f),(d,g),(d,h),$$
$$(e,f),(e,g),(e,h)\}.$$

Hence, the members of $\big(\{a,b\}\times\{f,g,h\}\big)\cup\big(\{c,d,e\}\times\{f,g,h\}\big)$ are the 15 ordered pairs shown in the preceding display. Note that the answer here is identical to that in part (c). This fact is due to the identity

$$(A\cup B)\times C=(A\times C)\cup(B\times C).$$

1.25 Answers will vary. One possibility is to take $I_n=(n-1,n]$ for $n\in\mathcal{Z}$. Clearly, these intervals constitute a countably infinite collection of intervals of $\mathcal{R}$, each of length 1. Also, suppose that $j,k\in\mathcal{Z}$ with $j\neq k$, say, $j<k$. If $x\in I_j$, then $x\le j\le k-1$ and, hence, $x\notin I_k$. Consequently, we see that $I_j\cap I_k=\emptyset$. Now, let $x\in\mathcal{R}$ and set $n=\min\{j\in\mathcal{Z}:x\le j\}$. Then $n-1<x\le n$ so that $x\in I_n$. Therefore, we have shown that

$$\mathcal{R}=\bigcup_{n\in\mathcal{Z}}I_n=\bigcup_{n=-\infty}^{\infty}(n-1,n].$$

Theory Exercises

1.29

a) By definition, $A\cup\emptyset$ consists of all elements that are either in A or $\emptyset$. But, $\emptyset$ contains no elements. Hence, we must have $A\cup\emptyset=A$.

b) If $x\in A$, then $x\in A$ or $x\in B$, which means that $x\in A\cup B$. Hence, $A\subset A\cup B$.

c) Suppose that $A=A\cup B$. Then, from part (b),

$$B\subset B\cup A=A\cup B=A.$$

Hence, $B\subset A$. Conversely, suppose that $B\subset A$. Let $x\in A\cup B$. Then either $x\in A$ or $x\in B$. However, as $B\subset A$, if $x\in B$, then $x\in A$. Hence, in either case, we have $x\in A$. Thus, we have shown that $A\cup B\subset A$. From part (b), $A\subset A\cup B$. Consequently, we have $A=A\cup B$.

1.33 We have $x\in B\cap\big(\bigcup_n A_n\big)$ iff $x\in B$ and $x\in\bigcup_n A_n$ iff $x\in B$ and $x\in A_n$ for some n iff $x\in B\cap A_n$ for some n iff $x\in\bigcup_n(B\cap A_n)$. Hence, Proposition 1.5(a) holds. Applying that result to the sets B^c and $A_1^c, A_2^c,\ldots$ and using De Morgan's laws, we get

$$B\cup\Big(\bigcap_n A_n\Big)=(B^c)^c\cup\bigg(\Big(\bigcap_n A_n\Big)^c\bigg)^c=\bigg(B^c\cap\Big(\bigcap_n A_n\Big)^c\bigg)^c=\bigg(B^c\cap\Big(\bigcup_n A_n^c\Big)\bigg)^c$$
$$=\Big(\bigcup_n(B^c\cap A_n^c)\Big)^c=\bigcap_n(B^c\cap A_n^c)^c=\bigcap_n\big((B^c)^c\cup(A_n^c)^c\big)$$
$$=\bigcap_n(B\cup A_n).$$

Hence, Proposition 1.5(b) holds.

Advanced Exercises

1.37 Define $f:\mathcal{N}^2\to\mathcal{N}$ by $f(m,n)=2^{m-1}(2n-1)$. We claim that f is one-to-one and onto $\mathcal{N}$, which will show that $\mathcal{N}^2$ is countably infinite. Let $k\in\mathcal{N}$. If k is even, let j denote the largest positive integer such that 2^j divides k. Then $k = 2^j(2n-1)$ for some $n \in \mathcal{N}$, and we have

$$f(j+1, n) = 2^{(j+1)-1}(2n-1) = 2^j(2n-1) = k.$$

If k is odd, then $(k+1)/2 \in \mathcal{N}$, and we have

$$f\big(1, (k+1)/2\big) = 2^{1-1}\left(2 \cdot \frac{k+1}{2} - 1\right) = 1 \cdot \big((k+1) - 1\big) = k.$$

Thus, we have shown that f is onto $\mathcal{N}$. Next, we establish that f is one-to-one. Suppose then that $f(m,n) = f(j,k)$, where, say, $m \geq j$. Then, we have $2^{m-1}(2n-1) = 2^{j-1}(2k-1)$ or, equivalently, $2^{m-j}(2n-1) = 2k-1$. As $2k-1$ is odd, we must have $m - j = 0$, or $m = j$, in which case, we have $2n - 1 = 2k - 1$ and, hence, $n = k$. Therefore, $(m,n) = (j,k)$.

1.41 Let A and B be two countable sets. If either A or B is empty, then $A \times B = \emptyset$ and, hence, is countable. Therefore, let us assume that both A and B are nonempty. By Exercise 1.38, each of A and B is the range of an infinite sequence, say, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, respectively. Define $f\colon \mathcal{N}^2 \to A \times B$ by $f(m,n) = (a_m, b_n)$. From Exercise 1.37, we know that $\mathcal{N}^2$ is countable and, clearly, f is onto $A \times B$. Thus, $A \times B$ is the image of $\mathcal{N}^2$ under f. Applying Exercise 1.39, we conclude that $A \times B$ is countable.

We next use mathematical induction to prove that the Cartesian product of a finite number of countable sets is countable. We have just verified that result for $n = 2$. Assuming its truth for $n - 1$, we now prove it for n. Thus, let $A_1, \ldots, A_n$ be countable sets. For convenience, set $B_k = \times_{j=1}^{k} A_j$ for each $k \in \mathcal{N}$. From the induction assumption, we know that B_{n-1} is countable and, hence, so is $B_{n-1} \times A_n$. Let us define $f\colon B_{n-1} \times A_n \to B_n$ by $f\big((a_1, \ldots, a_{n-1}), a_n\big) = (a_1, \ldots, a_{n-1}, a_n)$. Clearly, f is one-to-one and onto and, hence, as $B_{n-1} \times A_n$ is countable, so is $B_n = \times_{j=1}^{n} A_j$.

1.45

a) Suppose that I is finite, say, $I = \{1, 2, \ldots, N\}$. On the one hand, by Definition 1.5, $\times_{n=1}^{N} A_n$ is the set of all ordered N-tuples $(a_1, a_2, \ldots, a_N)$, where $a_n \in A_n$ for $1 \leq n \leq N$. On the other hand, by the definition of the Cartesian product of an indexed collection, $\times_{n=1}^{N} A_n$ is the set of all functions x on $\{1, 2, \ldots, N\}$ such that $x(n) \in A_n$ for $1 \leq n \leq N$. Identifying each such function x with the ordered N-tuple $\big(x(1), x(2), \ldots, x(N)\big)$, we obtain a one-to-one correspondence between the members of the Cartesian product $\times_{n=1}^{N} A_N$ as defined by Definition 1.5 and the members of the Cartesian product $\times_{n=1}^{N} A_n$ of the indexed collection $\{A_\iota\}_{\iota \in I}$.

b) False, it is not necessarily the case that the Cartesian product of a countable number of countable sets is countable. For instance, let D denote the set of decimal digits and let $E = D^{\infty} = D \times D \times D \times \cdots$, the Cartesian product of a countable number of countable (actually, finite) sets. Define $f\colon E \to [0,1]$ by $f(d_1, d_2, \ldots) = \sum_{n=1}^{\infty} d_n 10^{-n}$. We have $f(E) = [0,1]$; that is, $[0,1]$ is the image of f under E. If E were countable, then, by Exercise 1.39, so would be $[0,1]$. However, by Exercise 1.43(d), we know that $[0,1]$ is uncountable. Hence, E must be uncountable.

Review Exercises for Chapter 1

Basic Exercises

1.46

a) Referring to Table 1.1 on page 4, we see that 16 of the states are in the South and, of those, 11 seceded from the Union. Therefore, the probability that a randomly selected state in the South seceded from the union is 11/16, or 0.6875.

b) The population under consideration consists of the 16 states in the South.

1.47 From the table, the total number of winners for the years 1901–1997 is $190 + 71 + \cdots + 87 = 448$.
a) The number of winners from Japan is 4. Hence, the probability that the recipient selected is from Japan is $4/448 \approx 0.00893$.
b) The number of winners from either France or Germany is $25 + 61 = 86$. Hence, the probability that the recipient selected is from either France or Germany is $86/448 \approx 0.192$.
c) The number of winners from any country other than the United States is $448 - 190 = 258$. Hence, the probability that the recipient selected is from any country other than the United States is $258/448 \approx 0.576$.

1.48
a) Referring to the frequentist interpretation of probability on page 5, we see that, in a large number of tosses of the die, the result will be 3 about 1/6 of the time.
b) Again referring to the frequentist interpretation of probability, we see that, in a large number of tosses of the die, the result will be 3 or more about 2/3 of the time.
c) From part (a), we see that, in 10,000 tosses, the die will come up 3 about $10{,}000 \cdot (1/6)$ times, or roughly 1667 times.
d) From part (b), we see that, in 10,000 tosses, the die will come up 3 or more about $10{,}000 \cdot (2/3)$ times, or roughly 6667 times.

1.49 Consider the random experiment of selecting one voter (at random) from the population and let E denote the event that the voter chosen will vote *yes* on the proposition. Because 60% of the voters will vote *yes,* we know that $P(E) = 0.6$. Now, choosing n voters at random with replacement is equivalent to independently repeating the random experiment n times. Hence, from the frequentist interpretation of probability, specifically Relation (1.1) on page 5, we have, for large n, that

$$n(Y) = n(E) \approx P(E) \cdot n = 0.6n.$$

1.50
a) Squaring each element of the set $S = \{-2, -1, 0, 1, 2\}$, we find that the elements of $\{\, x^2 : x \in S \,\}$ are 0, 1, and 4.
b) If $-2 < x < 2$, then $0 \le x^2 < 4$ and, vice-versa. Hence, $\{\, x^2 : -2 < x < 2 \,\} = [0, 4)$.

1.51 The eight members of $\{0, 1\}^3$ are (0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), and (1, 1, 1). By associating 0 with a tail and 1 with a head, we can regard the eight members of $\{0, 1\}^3$ as the eight possible outcomes of the experiment of tossing a coin three times. We now consider the eight members of $\{0, 1\}^3$ a finite population. If we select a member at random from that population, then each member is equally likely to be the one obtained. Hence, that random experiment can be regarded as tossing a balanced coin three times. Note that there are three members of the population corresponding to getting two heads and one tail, namely, (0, 1, 1), (1, 0, 1), and (1, 1, 0). Therefore, the probability that you get two heads and one tail when you toss a balanced coin three times equals the probability that a member selected at random from $\{0, 1\}^3$ consists of exactly two 1s and one 0, which is 3/8, or 0.375.

1.52 We can proceed in several ways to obtain the simplification. One way is to apply the distributive law of Proposition 1.2(a) on page 12, as follows:

$$(A \cap B) \cup (A \cap B^c) = \big(A \cap (B \cup B^c)\big) = A \cap U = A.$$

1.53 For each $n \in \mathcal{N}$, let $A_n = \big(1/(n+1), 1/n\big]$ and $B_n = \big[1/(n+1), 1/n\big]$.
a) Set $A = \bigcup_{n=1}^{\infty} A_n$. If $x \in A$, then there is an $n \in \mathcal{N}$ such that $x \in A_n$ and, so, $1/(n+1) < x \le 1/n]$. But then we have

$$0 < \frac{1}{n+1} < x \le \frac{1}{n} \le 1.$$

Thus, $A \subset (0, 1]$. Conversely, suppose that $x \in (0, 1]$. Then $x > 0$. Let n be the smallest positive integer such that $1/(n+1) < x$. Then $1/(n+1) < x \le 1/n$ so that $x \in A_n$, which, in turn, implies that $x \in A$. Thus, $(0, 1] \subset A$. We have shown that $A \subset (0, 1]$ and $(0, 1] \subset A$. Consequently, $A = (0, 1]$.
b) Set $B = \bigcup_{n=1}^{\infty} B_n$. If $x \in B$, then there is an $n \in \mathcal{N}$ such that $x \in B_n$ and, so, $1/(n+1) \le x \le 1/n]$. But then we have

$$0 < \frac{1}{n+1} \le x \le \frac{1}{n} \le 1.$$

Thus, $B \subset (0, 1]$. Conversely, suppose that $x \in (0, 1]$. Then $x > 0$. Let n be the smallest positive integer such that $1/(n+1) < x$. Then $1/(n+1) < x \le 1/n$ so that $x \in B_n$, which, in turn, implies that $x \in B$. Thus, $(0, 1] \subset B$. We have shown that $B \subset (0, 1]$ and $(0, 1] \subset B$. Consequently, $B = (0, 1]$.
c) The sets in the union in part (a) are pairwise disjoint. Indeed, let $m \ne n$, say, $m < n$. Then, if $x \in A_n$, we have $x \le 1/n \le 1/(m+1)$ and, hence, $x \notin A_m$. Consequently, $A_n \cap A_m = \emptyset$. The sets in the union in part (b) are not pairwise disjoint. For instance, we have

$$B_1 \cap B_2 = [1/2, 1] \cap [1/3, 1/2] = \{1/2\} \ne \emptyset.$$

However,

$$\bigcap_{n=1}^{\infty} B_n \subset B_1 \cap B_3 = [1/2, 1] \cap [1/4, 1/3] = \emptyset.$$

Consequently, $\bigcap_{n=1}^{\infty} B_n = \emptyset$.

1.54 You may find it helpful to refer to the discussion of finite and infinite sets on pages 19 and 20. Also, note that other methods can be used to obtain the following results.
a) We have

$$\{1, 2, 3\} \times \{2, 3, 4\} = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}.$$

Thus, $\{1, 2, 3\} \times \{2, 3, 4\}$ consists of nine elements and, hence, is finite.
b) Let $A = \{1, 2, 3\} \times [2, 4]$. We have

$$A = \{(1, y) : y \in [2, 4]\} \cup \{(2, y) : y \in [2, 4]\} \cup \{(3, y) : y \in [2, 4]\}.$$

Denote by B the first set in the union of the preceding display. Define $f: [2, 4] \to B$ by $f(y) = (1, y)$. Clearly, f is one-to-one and onto. Hence, from Exercise 1.43, we deduce that B is uncountable. As $B \subset A$, it follows from Exercise 1.40 that A is uncountable.
c) Let A be as in part (b) and set $C = [2, 4] \times \{1, 2, 3\}$. Define $f: A \to C$ by $f(x, y) = (y, x)$. Clearly, f is one-to-one and onto. From part (b), we know that A is uncountable; hence, so is C.
d) Let $D = \{1, 2, 3\} \times \{1, 2, 3, \ldots\}$. From Exercise 1.41, we see that D is countable, being the Cartesian product of two countable sets (one finite and the other countably infinite). Obviously, D is infinite and, therefore, we conclude that it is countably infinite.
e) Let $E = [1, 3] \times [2, 4]$ and let B be as in part (b). As we have seen, B is uncountable. Therefore, because $E \supset B$, we deduce from Exercise 1.40 that E is uncountable.
f) We note that $\bigcup_{n=1}^{\infty} \{n, n+1, n+2\} = \mathcal{N}$, which is countably infinite.
g) Let $A_n = \{n, n+1, n+2\}$ and $F = \bigcap_{n=1}^{\infty} A_n$. We have

$$F \subset A_1 \cap A_4 = \{1, 2, 3\} \cap \{4, 5, 6\} = \emptyset.$$

Therefore, $F = \emptyset$ and, hence, is finite.

1.55
a) From De Morgan's laws,

$$\left(\{3, 4\}^c \cap \{4, 5\}^c\right)^c = \left(\{3, 4\}^c\right)^c \cup \left(\{4, 5\}^c\right)^c = \{3, 4\} \cup \{4, 5\} = \{3, 4, 5\}.$$

b) We have

$$\big(\{3,4\}^c \cap \{4,5\}^c\big)^c = \big(\{\ldots,-2,-1,0,1,2,5,6,\ldots\} \cap \{\ldots,-2,-1,0,1,2,3,6,7,\ldots\}\big)^c$$
$$= \big(\{\ldots,-2,-1,0,1,2,6,7,\ldots\}\big)^c = \{3,4,5\}.$$

Theory Exercises

1.56 Let $B_1 = A_1$ and, for $k \geq 2$, set

$$B_k = A_k \cap \left(\bigcup_{j=1}^{k-1} A_j\right)^c = A_1^c \cap \cdots \cap A_{k-1}^c \cap A_k.$$

Observe that, for all $k \in \mathcal{N}$, we have $B_k \subset A_k$ and $B_k \subset A_j^c$ for $1 \leq j \leq k-1$. Let $m \neq n$, say, $m < n$. Then $B_m \subset A_m$ and $B_n \subset A_m^c$, so that

$$B_m \cap B_n \subset A_m \cap A_m^c = \emptyset.$$

Consequently, $B_m \cap B_n = \emptyset$, and we see that $B_1, B_2, \ldots$ are pairwise disjoint sets. We use these sets in parts (a) and (b).

a) For convenience, set $A^{(n)} = \bigcup_{j=1}^n A_j$ and $B^{(n)} = \bigcup_{j=1}^n B_j$. As $B_j \subset A_j$ for all $j \in \{1, 2, \ldots, n\}$, we have $B^{(n)} \subset A^{(n)}$. Conversely, suppose that $x \in A^{(n)}$. Then $x \in A_j$ for some $j \in \{1, 2, \ldots, n\}$. Let k be the smallest such j. Then $x \in A_k$ and $x \notin A_j$ for $1 \leq j \leq k-1$, which means that $x \in B_k$ and, consequently, that $x \in B^{(n)}$. Therefore, $A^{(n)} \subset B^{(n)}$. We have now shown that $B^{(n)} \subset A^{(n)}$ and $A^{(n)} \subset B^{(n)}$. Hence, $A^{(n)} = B^{(n)}$.

b) We use the notation of part (a) and recall that $A^{(n)} = B^{(n)}$ for all $n \in \mathcal{N}$. Let $A = \bigcup_{n=1}^\infty A_n$ and $B = \bigcup_{n=1}^\infty B_n$. We observe that $A = \bigcup_{n=1}^\infty A^{(n)}$ and $B = \bigcup_{n=1}^\infty B^{(n)}$. Hence,

$$A = \bigcup_{n=1}^\infty A^{(n)} = \bigcup_{n=1}^\infty B^{(n)} = B.$$

1.57

a) From the distributive law of Proposition 1.2(a) on page 12 and the fact that $A \cap C \subset A$, we get

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C) \subset A \cup (B \cap C).$$

b) Answers will vary. Note, however, from the solution to part (a) that any choice of A, B, and C in which $A \cap C \neq A$ will do the trick. For instance, take $A = \{1\}$ and $B = C = \emptyset$. Then

$$(A \cup B) \cap C = (\{1\} \cup \emptyset) \cap \emptyset = \emptyset \neq \{1\} = \{1\} \cup \emptyset = \{1\} \cup (\emptyset \cap \emptyset) = A \cup (B \cap C).$$

c) If $A \subset C$, then $A \cap C = A$. Hence, from the distributive law of Proposition 1.2(a),

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C) = A \cup (B \cap C).$$

d) If $(A \cup B) \cap C = A \cup (B \cap C)$, then

$$A \subset A \cup (B \cap C) = (A \cup B) \cap C \subset C.$$

1.58

a) Let $\mathcal{C} = \{\, C \subset U : A \subset C \text{ and } B \subset C \,\}$. We want to prove that $A \cup B = \bigcap_{C \in \mathcal{C}} C$. To begin, we note that, because $A \subset A \cup B$ and $B \subset A \cup B$, we have $A \cup B \in \mathcal{C}$. Thus, $A \cup B \supset \bigcap_{C \in \mathcal{C}} C$. Conversely, let $C \in \mathcal{C}$. Then $A \subset C$ and $B \subset C$, which implies that $A \cup B \subset C$. Therefore, $A \cup B \subset \bigcap_{C \in \mathcal{C}} C$.

b) If $A \subset D$ and $B \subset D$, then $A \cup B \subset D$. Hence, we see that $A \cup B$ is the smallest set that contains both A and B as subsets.
c) Let $\mathcal{D} = \{ C \subset U : A \supset C \text{ and } B \supset C \}$. We want to prove that $A \cap B = \bigcup_{C \in \mathcal{D}} C$. To begin, we note that, because $A \supset A \cap B$ and $B \supset A \cap B$, we have $A \cap B \in \mathcal{D}$. Thus, $A \cap B \subset \bigcup_{C \in \mathcal{D}} C$. Conversely, let $C \in \mathcal{D}$. Then $A \supset C$ and $B \supset C$, which implies that $A \cap B \supset C$. Therefore, $A \cap B \supset \bigcup_{C \in \mathcal{D}} C$.
d) If $A \supset D$ and $B \supset D$, then $A \cap B \supset D$. Hence, we see that $A \cap B$ is the largest set that is contained as a subset by both A and B.
e) Let $A_1, A_2, \ldots$ be a countable sequence of subsets of U.

Generalization of part (a): Let $\mathcal{E} = \{ C \subset U : A_n \subset C \text{ for all } n \}$. We want to prove that

$$\bigcup_n A_n = \bigcap_{C \in \mathcal{E}} C.$$

For convenience, set $A = \bigcup_n A_n$. Because $A_n \subset A$ for all n, we have $A \in \mathcal{E}$. Thus, $A \supset \bigcap_{C \in \mathcal{E}} C$. Conversely, let $C \in \mathcal{E}$. Then $A_n \subset C$ for all n, which implies that $A \subset C$. Therefore, $A \subset \bigcap_{C \in \mathcal{E}} C$.

Generalization of part (b): If $A_n \subset D$ for all n, then $\bigcup_n A_n \subset D$. Hence, we see that $\bigcup_n A_n$ is the smallest set that contains all A_ns as subsets.

Generalization of part (c): Let $\mathcal{F} = \{ C \subset U : A_n \supset C \text{ for all } n \}$. We want to prove that

$$\bigcap_n A_n = \bigcup_{C \in \mathcal{F}} C.$$

For convenience, set $A = \bigcap_n A_n$. Because $A_n \supset A$ for all n, we have $A \in \mathcal{F}$. Thus, $A \subset \bigcup_{C \in \mathcal{F}} C$. Conversely, let $C \in \mathcal{F}$. Then $A_n \supset C$ for all n, which implies that $A \supset C$. Therefore, $A \supset \bigcup_{C \in \mathcal{F}} C$.

Generalization of part (d): If $A_n \supset D$ for all n, then $\bigcap_n A_n \supset D$. Hence, we see that $\bigcap_n A_n$ is the largest set that is contained as a subset of all A_ns.

1.59
a) This result is true. Indeed, we have $(x, y) \in A \times (B \cup C)$ iff $x \in A$ and $y \in B \cup C$ iff $x \in A$ and either $y \in B$ or $y \in C$ iff either $x \in A$ and $y \in B$ or $x \in A$ and $y \in C$ iff either $(x, y) \in A \times B$ or $(x, y) \in A \times C$ iff $(x, y) \in (A \times B) \cup (A \times C)$. Hence, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
b) This result is true. Indeed, we have $(x, y) \in A \times (B \cap C)$ iff $x \in A$ and $y \in B \cap C$ iff $x \in A$ and $y \in B$ and $y \in C$ iff $x \in A$ and $y \in B$ and $x \in A$ and $y \in C$ iff $(x, y) \in A \times B$ and $(x, y) \in A \times C$ iff $(x, y) \in (A \times B) \cap (A \times C)$. Hence, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
c) This result is not always true. For instance, let $A = \{0\}$ and $B = \{1\}$. Then

$$A \times B = \{(0, 1)\} \neq \{(1, 0)\} = B \times A.$$

Advanced Exercises

1.60 Refer to the table in the problem statement.
a) The number of left-handed individuals is $71 + 92 = 163$. Hence, the probability that the person obtained is left-handed is $163/525 \approx 0.310$.
b) The number of colorblind individuals who are not ambidextrous is $71 + 61 = 132$. Hence, the probability that the person obtained is colorblind but not ambidextrous is $132/525 \approx 0.251$.
c) The number of left-handed individuals is $71 + 92 = 163$ of which 71 are colorblind. Hence, if the person selected is left-handed, the probability that he or she is colorblind is $71/163 \approx 0.436$.
d) The number of colorblind individuals is $71 + 61 + 37 = 169$ of which 71 are left-handed. Hence, if the person selected is colorblind, the probability that he or she is left-handed is $71/169 \approx 0.420$.

e) The number of left-handed, colorblind women is 24. Hence, the probability that the person obtained is a left-handed, colorblind woman is $24/525 \approx 0.0457$.

1.61 The probability is 0.75 that a randomly selected adult female believes that having a "cyber affair" is cheating. Therefore, the odds against an adult female believing that having a "cyber affair" is cheating are

$$1 - 0.75 \text{ to } 0.75 \qquad \text{or} \qquad 0.25 \text{ to } 0.75 \qquad \text{or} \qquad 1 \text{ to } 3.$$

1.62
a) We have

$$B^c = \{\, x : x \notin B \,\} = \{\, x : x \in U \text{ and } x \notin B \,\} = U \setminus B.$$

b) We have $x \in A \setminus B$ iff $x \in A$ and $x \notin B$ iff $x \in A$ and $x \in B^c$ iff $x \in A \cap B^c$. Consequently, we have shown that $A \setminus B = A \cap B^c$.
c) Applying part (b) and De Morgan's law of Proposition 1.1(b) on page 11, we get

$$(A \setminus B)^c = \left(A \cap B^c\right)^c = A^c \cup \left(B^c\right)^c = A^c \cup B.$$

1.63 We note that, by definition, $x \in A \,\triangle\, B$ iff either $x \in A$ or $x \in B$, and $x \notin A \cap B$, which is the case iff $x \in A \cup B$ and $x \in (A \cap B)^c$ iff $x \in (A \cup B) \cap (A \cap B)^c$. Thus, $A \,\triangle\, B = (A \cup B) \cap (A \cap B)^c$. Applying De Morgan's laws, the distributive laws, and Exercise 1.62(b), we get

$$\begin{aligned} A \,\triangle\, B &= (A \cup B) \cap (A \cap B)^c = (A \cup B) \cap \left(A^c \cup B^c\right) = \left((A \cup B) \cap A^c\right) \cup \left((A \cup B) \cap B^c\right) \\ &= \left((A \cap A^c) \cup (B \cap A^c)\right) \cup \left((A \cap B^c) \cup (B \cap B^c)\right) = \emptyset \cup (B \setminus A) \cup (A \setminus B) \cup \emptyset \\ &= (A \setminus B) \cup (B \setminus A). \end{aligned}$$

1.64
a) We note that $x \in B \,\triangle\, C$ iff x is a member of exactly one of B and C. Hence, $x \notin B \,\triangle\, C$ iff either x is in both B and C or x is in neither B nor C. In other words,

$$(B \,\triangle\, C)^c = (B \cap C) \cup (B^c \cap C^c),$$

a result that we could also obtain by applying properties of set operations and results from Exercises 1.62 and 1.63. Referring now to those two exercises, we get

$$\begin{aligned} A \,\triangle\, (B \,\triangle\, C) &= \left(A \cap (B \,\triangle\, C)^c\right) \cup \left((B \,\triangle\, C) \cap A^c\right) \\ &= \Big(A \cap \left((B \cap C) \cup (B^c \cap C^c)\right)\Big) \cup \Big(\left((B \cap C^c) \cup (C \cap B^c)\right) \cap A^c\Big) \\ &= \left(A \cap B \cap C\right) \cup \left(A \cap B^c \cap C^c\right) \cup \left(B \cap C^c \cap A^c\right) \cup \left(C \cap B^c \cap A^c\right). \end{aligned}$$

Replacing A by C, B by A, and C by B in the previous display, we find that $C \,\triangle\, (A \,\triangle\, B) = A \,\triangle\, (B \,\triangle\, C)$. The required result now follows from the easily established fact that $E \,\triangle\, F = F \,\triangle\, E$.
b) We have

$$A \,\triangle\, U = \left(A \cap U^c\right) \cup \left(U \cap A^c\right) = (A \cap \emptyset) \cup A^c = \emptyset \cup A^c = A^c.$$

c) We have

$$A \,\triangle\, \emptyset = \left(A \cap \emptyset^c\right) \cup \left(\emptyset \cap A^c\right) = (A \cap U) \cup \emptyset = A \cup \emptyset = A.$$

d) We have

$$A \,\triangle\, A = \left(A \cap A^c\right) \cup \left(A \cap A^c\right) = \emptyset \cup \emptyset = \emptyset.$$

1.65

a) Applying properties of set operations, we get

$$\begin{aligned} A \cap (B \triangle C) &= A \cap \big((B \cap C^c) \cup (C \cap B^c)\big) = \big(A \cap (B \cap C^c)\big) \cup \big(A \cap (C \cap B^c)\big) \\ &= \big(A \cap B \cap C^c\big) \cup \big(A \cap C \cap B^c\big) = \big((A \cap B) \cap (A^c \cup C^c)\big) \cup \big((A \cap C) \cap (A^c \cup B^c)\big) \\ &= \big((A \cap B) \cap (A \cap C)^c\big) \cup \big((A \cap C) \cap (A \cap B)^c\big) = (A \cap B) \triangle (A \cap C). \end{aligned}$$

b) Again applying properties of set operations, we get

$$\begin{aligned} (A \cup B) \triangle (A \cup C) &= \big((A \cup B) \cap (A \cup C)^c\big) \cup \big((A \cup C) \cap (A \cup B)^c\big) \\ &= \big((A \cup B) \cap (A^c \cap C^c)\big) \cup \big((A \cup C) \cap (A^c \cap B^c)\big) \\ &= \big(B \cap A^c \cap C^c\big) \cup \big(C \cap A^c \cap B^c\big) = A^c \cap \big((B \cap C^c) \cup (C \cap B^c)\big) \\ &= A^c \cap (B \triangle C). \end{aligned}$$

Therefore,

$$(A \cup B) \triangle (A \cup C) = A^c \cap (B \triangle C) \subset B \triangle C \subset A \cup (B \triangle C).$$

Thus, we have shown that $(A \cup B) \triangle (A \cup C) \subset A \cup (B \triangle C)$.

c) Suppose that $A = \emptyset$. Then

$$A \cup (B \triangle C) = \emptyset \cup (B \triangle C) = B \triangle C = (\emptyset \cup B) \triangle (\emptyset \cup C) = (A \cup B) \triangle (A \cup C).$$

Conversely, suppose that $A \cup (B \triangle C) = (A \cup B) \triangle (A \cup C)$. Then, referring to the solution to part (b), we find that

$$A \subset A \cup (B \triangle C) = (A \cup B) \triangle (A \cup C) = A^c \cap (B \triangle C) \subset A^c.$$

From this result, we conclude that $A = A \cap A \subset A \cap A^c = \emptyset$; that is, $A = \emptyset$. We have therefore shown that $A \cup (B \triangle C) = (A \cup B) \triangle (A \cup C)$ precisely when $A = \emptyset$.

1.66 Suppose that $B = \emptyset$. Then, from Exercise 1.64(c), we have $A \triangle B = A \triangle \emptyset = A$. Conversely, suppose that $A = A \triangle B$. Then, applying in turn parts (c), (d), (a), and (d) of Exercise 1.64, we find that

$$B = B \triangle \emptyset = \emptyset \triangle B = (A \triangle A) \triangle B = A \triangle (A \triangle B) = A \triangle A = \emptyset.$$